Plasma Physics and Dynamo Theory • Module 14

Shocks: hydrodynamic, MHD and collisionless

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Outline

1. Why shocks form: nonlinear steepening
2. Rankine–Hugoniot conditions: compression, entropy, blast waves
3. MHD shocks: coplanarity; fast, slow, intermediate; switch-on and switch-off
4. Collisionless shocks: the Earth's bow shock
5. Diffusive shock acceleration: the spectrum of cosmic rays

Reading

Zel'dovich & Raizer, ch. I
Treumann (2009) on collisionless shocks
Drury (1983) on shock acceleration

Notes: chapter 14 (with more on the shock structure, the perpendicular shock and the acceleration time)

Where we are

- module 12a: ideal MHD in conservation form, $\partial_t \boldsymbol{q} + \nabla \cdot \mathrm{F} = 0$ for mass, momentum, energy and $\boldsymbol{B}$
- module 12b: *small* perturbations travel at c_s, v_A and the fast and slow speeds v_f, v_s
- module 4b: Fermi acceleration with gain ξ and escape probability P per cycle gives $N(E) \propto E^{-s}$, $s \simeq 1 + P/\xi$

Today: large perturbations. A compressive wave steepens until it becomes a discontinuity, a shock. Three questions:

1. what jumps across it, and by how much? (conservation laws)
2. how does a plasma without collisions dissipate? (collisionless shocks)
3. what does a shock do to fast particles? (the origin of cosmic rays)

Why shocks form

A simple wave: each value of u moves at its own speed

One-dimensional isentropic gas, $p \propto \rho^\gamma$, $c^2 = \mathrm{d}p/\mathrm{d}\rho = \gamma p/\rho$:

$$\partial_t \rho + u\,\partial_x \rho + \rho\,\partial_x u = 0, \qquad \partial_t u + u\,\partial_x u + \frac{c^2}{\rho}\,\partial_x \rho = 0$$

A wave running into gas at rest: try $u = u(\rho)$, so $\partial u = u'\,\partial\rho$. The two equations become

$$\partial_t \rho + (u + \rho u')\,\partial_x \rho = 0, \qquad \partial_t \rho + \left(u + \frac{c^2}{\rho u'}\right)\partial_x \rho = 0$$

Consistent only if $\rho u' = c^2/(\rho u')$, i.e. $u' = c/\rho$ (wave moving to $+x$). Each value of ρ is carried at speed $u + c$.

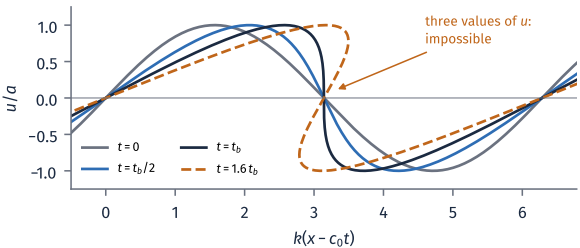
Since $c \propto \rho^{(\gamma-1)/2}$: $\mathrm{d}c = \frac{\gamma-1}{2}\,c\,\frac{\mathrm{d}\rho}{\rho} = \frac{\gamma-1}{2}\,\mathrm{d}u$, so $c = c_0 + \frac{\gamma-1}{2}u$ and

Key result

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u + c = c_0 + \frac{\gamma+1}{2}\,u \qquad \text{(Riemann 1860)}$$

Steepening and breaking

Two effects add up: a compressed parcel moves *with* the flow (u), and it is *hotter*, so sound is faster ($\frac{\gamma-1}{2}u$). Crests overtake troughs.



Initial $u = a \sin k x_0$. In the frame moving at c_0 the value at x_0 is found at $x = x_0 + \frac{\gamma+1}{2}\, a \sin(k x_0)\, t$. Two values meet when $\partial x / \partial x_0 = 1 + \frac{\gamma+1}{2}\, k a \cos(k x_0)\, t = 0$, first at $\cos k x_0 = -1$:

$$t_b = \frac{2}{(\gamma+1)\, k a}$$

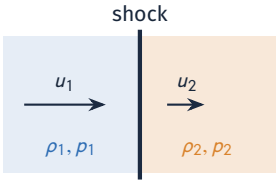
After t_b the solution would be multivalued. In reality the gradients grow until viscosity and heat conduction act: a thin layer, a few mean free paths thick, the shock. Outside it, the ideal equations hold.

Rankine–Hugoniot conditions

Conservation across a thin layer

In the frame of the shock the flow is steady: gas enters at u_1 (1, upstream) and leaves at u_2 (2, downstream). Conservation form, $\partial_t q + \partial_x F = 0$, with $\partial_t = 0$:

$$\frac{\mathrm{d}F}{\mathrm{d}x} = 0 \quad \Rightarrow \quad [F] \equiv F_2 - F_1 = 0$$



whatever happens inside the layer: viscous and conductive fluxes vanish where the gradients vanish.

Mass, momentum and energy fluxes (module 12a with $\boldsymbol{B} = 0$), enthalpy $h = \frac{\gamma}{\gamma-1}\frac{p}{\rho}$:

Key result

$$[\rho u] = 0, \qquad [\rho u^2 + p] = 0, \qquad \left[\rho u\left(\frac{u^2}{2} + h\right)\right] = 0$$

A shock moving at V_s into gas at rest has $u_1 = V_s$ (Rankine 1870, Hugoniot 1887).

The compression ratio

Unknowns ρ_2, u_2, p_2. Let $j = \rho_1 u_1 = \rho_2 u_2$, $r = \rho_2/\rho_1 = u_1/u_2$, $M_1^2 = u_1^2/c_1^2 = \rho_1 u_1^2/\gamma p_1$.
Momentum: $p_2 - p_1 = j(u_1 - u_2)$, so

$$\frac{p_2}{\rho_1 u_1^2} = \frac{1}{\gamma M_1^2} + 1 - x, \qquad x \equiv \frac{1}{r} = \frac{u_2}{u_1}$$

Energy, divided by $j u_1^2$: $\frac{1}{2} + \frac{1}{(\gamma-1)M_1^2} = \frac{x^2}{2} + \frac{\gamma}{\gamma-1}\, x\, \frac{p_2}{\rho_1 u_1^2}$. Insert p_2 and multiply by $2(\gamma-1)$:

$$(\gamma+1)\, x^2 - 2\left(\gamma + \frac{1}{M_1^2}\right) x + (\gamma - 1) + \frac{2}{M_1^2} = 0$$

$x = 1$ is a root (no jump). The product of the roots gives the other:

Key result

$$r = \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2}, \qquad \frac{p_2}{p_1} = \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1}$$

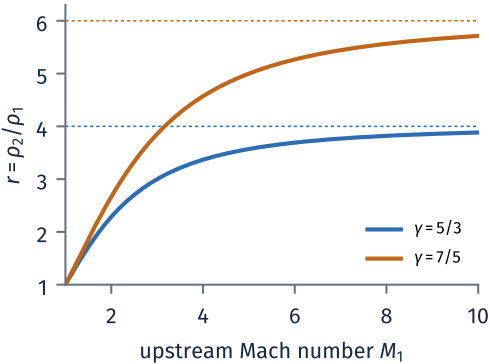
The strong-shock limit

$M_1 \to \infty$: the pressure grows without limit, the density does not:

$$r \to \frac{\gamma+1}{\gamma-1} = 4 \ (\gamma = 5/3), \qquad p_2 \to \frac{2}{\gamma+1}\rho_1 u_1^2$$

Temperature, $T_2 = \mu m_p p_2/\rho_2$ with $u_1 = V_s$:

$$T_2 = \frac{2(\gamma-1)}{(\gamma+1)^2}\mu m_p V_s^2 = \frac{3}{16}\mu m_p V_s^2$$



- the kinetic energy of the incoming flow becomes heat: $V_s = 1000$ km/s, $\mu = 0.6$ gives $T_2 = 1.2$ keV $= 1.4 \times 10^7$ K; a supernova shock at 5000 km/s gives 30 keV. Shocks make X-ray gas
- downstream the flow is subsonic; in the lab frame the gas behind moves at $\frac{3}{4}V_s$

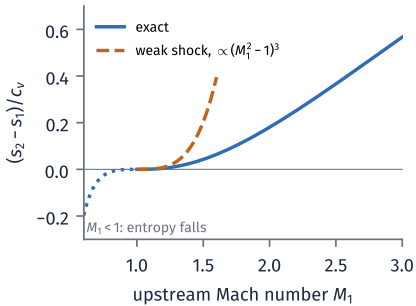
Entropy: why only compressive shocks

$s = c_v \ln(p/\rho^\gamma)$. With the two jumps p_2/p_1 and r:

$$\frac{s_2 - s_1}{c_v} = \ln \frac{p_2}{p_1} - \gamma \ln r(M_1)$$

Expand near $M_1 = 1$ (first and second orders cancel):

$$\frac{s_2 - s_1}{c_v} = \frac{2\gamma(\gamma-1)}{3(\gamma+1)^2} (M_1^2 - 1)^3 + \dots$$



- $M_1 < 1$ would give $r < 1$ and $s_2 < s_1$: an expansion shock violates the second law. Shocks are compressive: supersonic upstream, subsonic downstream
- weak shocks are nearly isentropic: almost sound waves
- the entropy jump is fixed by the conservation laws, not by the viscosity, which only sets the thickness. A plasma without collisions must produce the same entropy in another way

The Sedov–Taylor blast wave

Energy E released at a point in gas of density ρ. Once the swept-up mass dominates, and before radiation matters, $R(t)$ can depend only on E, ρ, t:

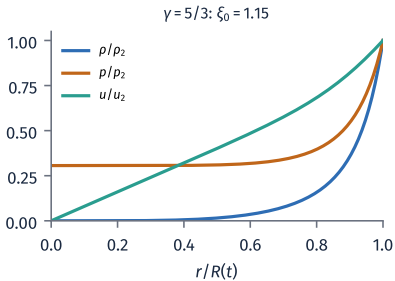
$$[E] = \mathrm{g\,cm^2\,s^{-2}}, \quad [\rho] = \mathrm{g\,cm^{-3}} \;\Rightarrow\; \left[\frac{E\,t^2}{\rho}\right] = \mathrm{cm^5}$$

The only length:

Key result

$$R = \xi_0 \left(\frac{E\,t^2}{\rho}\right)^{1/5}, \qquad V_s = \frac{\mathrm{d}R}{\mathrm{d}t} = \frac{2R}{5t}$$

ξ_0 is fixed by energy conservation inside the self-similar solution: $\xi_0 = 1.15$ for $\gamma = 5/3$ (Sedov 1946, Taylor 1950).



Self-similar profiles: the swept-up mass piles up in a thin shell behind the shock

Supernova remnants

A supernova: ejecta of $M_{\rm ej} \approx 1.4\,M_\odot$ with $E = 10^{51}$ erg, i.e. $(2E/M_{\rm ej})^{1/2} \approx 8500$ km/s, into gas with $n = 1$ cm^{-3} ($\rho = 1.4\,m_p n$).

1. **free expansion** until the swept-up mass equals $M_{\rm ej}$: $R \approx 2$ pc, $t \approx 250$ yr
2. **Sedov–Taylor** (the energy is conserved, the swept-up mass dominates)
3. **radiative**: when T_2 falls to $\sim 10^6$ K ($t \approx 3 \times 10^4$ yr) the shell cools: a snowplough

t	R	V_s	$T_2 = \frac{3}{16}\mu m_p V_s^2$
10^3 yr	5.0 pc	1900 km/s	4.5 keV
10^4 yr	12.5 pc	490 km/s	0.3 keV

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Age 1000 yr, $n \approx 0.05$ cm^{-3} (high above the Galactic plane): Sedov gives $R \approx 9$ pc, $V_s \approx 3500$ km/s. Observed: radius 15′ at 2.2 kpc, i.e. 9.6 pc.

MHD shocks

MHD jump conditions

Shock plane $\perp \hat{n} = \hat{x}$; u_n, B_n normal, $\boldsymbol{u}_t, \boldsymbol{B}_t$ tangential components. Take the x components of the fluxes of module 12a, and $\nabla \cdot \boldsymbol{B} = 0$:

$$[B_n] = 0, \qquad [\rho u_n] = 0$$

$$\left[\rho u_n^2 + p + \frac{B_t^2}{8\pi}\right] = 0 \qquad \text{(the } B_n^2 \text{ terms cancel since } [B_n] = 0\text{)}$$

$$\left[\rho u_n \boldsymbol{u}_t - \frac{B_n \boldsymbol{B}_t}{4\pi}\right] = 0 \qquad \text{(tangential momentum: tension)}$$

$$[u_n \boldsymbol{B}_t - B_n \boldsymbol{u}_t] = 0 \qquad \text{(induction; } c[\boldsymbol{E}_t] = 0\text{)}$$

$$\left[\rho u_n \left(\frac{u^2}{2} + h\right) + \frac{B^2 u_n - (\boldsymbol{u} \cdot \boldsymbol{B}) B_n}{4\pi}\right] = 0$$

Seven equations for $\rho_2, p_2, u_{n2}, \boldsymbol{u}_{t2}, \boldsymbol{B}_{t2}$ (de Hoffmann & Teller 1950).

The perpendicular shock

$B_n = 0$: no tension. Induction gives $[u_n B_t] = 0$, so $B_2/B_1 = u_1/u_2 = r$: the field is compressed with the gas.

- momentum: $\rho u^2 + p + B^2/8\pi$; magnetic pressure $\propto \rho^2$ behaves like a gas with $\gamma = 2$
- energy flux: $\rho u(u^2/2 + h) + uB^2/4\pi$

Eliminating p_2 as before (notes), with $\beta_1 = 8\pi p_1/B_1^2$:

$$2(2-\gamma)\, r^2 + \gamma\left[2\beta_1 + (\gamma-1)\beta_1 M_1^2 + 2\right] r - \gamma(\gamma+1)\beta_1 M_1^2 = 0$$

- $\beta_1 \to \infty$: the hydrodynamic $r(M_1)$; the positive root is > 1 only if $u_1 > v_{f1} = (c_1^2 + v_{A1}^2)^{1/2}$
- the field resists compression: r is smaller at given M_1, but still $r \to 4$ as $u_1/v_{f1} \to \infty$

The coplanarity theorem

Tangential momentum and induction, with $j = \rho u_n$ the same on both sides:

$$j\,[\boldsymbol{u}_t] = \frac{B_n}{4\pi}[\boldsymbol{B}_t], \qquad [u_n \boldsymbol{B}_t] = B_n[\boldsymbol{u}_t]$$

Eliminate $[\boldsymbol{u}_t]$: $[u_n \boldsymbol{B}_t] = \frac{B_n^2}{4\pi j}[\boldsymbol{B}_t]$, i.e. multiplying by j,

$$\left(j u_{n2} - \frac{B_n^2}{4\pi}\right)\boldsymbol{B}_{t2} = \left(j u_{n1} - \frac{B_n^2}{4\pi}\right)\boldsymbol{B}_{t1}$$

The brackets are scalars, so $\boldsymbol{B}_{t2}$ is a multiple of $\boldsymbol{B}_{t1}$. Their meaning: $j u_n - B_n^2/4\pi = \rho\,(u_n^2 - v_{An}^2)$, with $v_{An} = B_n/\sqrt{4\pi\rho}$ the Alfvén speed along the normal; the sign says whether the flow outruns an Alfvén wave.

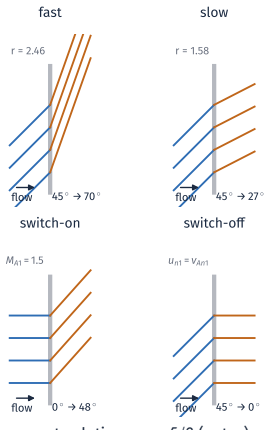
Key result

$\boldsymbol{B}_{t2} \parallel \boldsymbol{B}_{t1}$: $\boldsymbol{B}_1$, $\boldsymbol{B}_2$ and $\hat{n}$ lie in one plane, unless both brackets vanish ($u_n = v_{An}$ on both sides: then $\rho_2 = \rho_1$ and $\boldsymbol{B}_t$ just rotates, a rotational discontinuity).

Fast, slow and intermediate shocks

In $(ju_{n2} - B_n^2/4\pi)\boldsymbol{B}_{t2} = (ju_{n1} - B_n^2/4\pi)\boldsymbol{B}_{t1}$, j and B_n are the same on both sides, and $u_{n2} < u_{n1}$:

- both brackets > 0 ($u_n > v_{An}$ on both sides): $|\boldsymbol{B}_{t2}| > |\boldsymbol{B}_{t1}|$, the field bends away from the normal: fast shock. Both < 0: bends towards it: slow shock
- opposite signs: $\boldsymbol{B}_t$ reverses: intermediate shock (whether it is stable is still debated)
- **switch-off**: $u_{n1} = v_{An1}$, so $\boldsymbol{B}_{t2} = 0$. **Switch-on**: $\boldsymbol{B}_{t1} = 0$, $\boldsymbol{B}_{t2} \neq 0$ needs $u_{n2} = v_{An2}$, i.e. $M_{A1}^2 = r$: only for $1 < M_{A1} < 2$ ($\beta_1 \to 0$)



Collisionless shocks

Dissipation without collisions

Solar wind at 1 au: $n = 5$ cm^{-3}, $T = 10$ eV, $B = 5$ nT, $V = 400$ km/s.

collisional mean free path	$1.44 \times 10^{13}\, T_{\mathrm{eV}}^2/(n \ln \Lambda)$	≈ 0.8 au
ion inertial length	c/ω_{pi}	≈ 100 km
gyroradius of the incoming flow	V/Ω_i	≈ 840 km
observed bow-shock ramp		$\sim c/\omega_{pi}$

The shock is 10^6 times thinner than a mean free path: dissipation is by the fields, not by collisions.

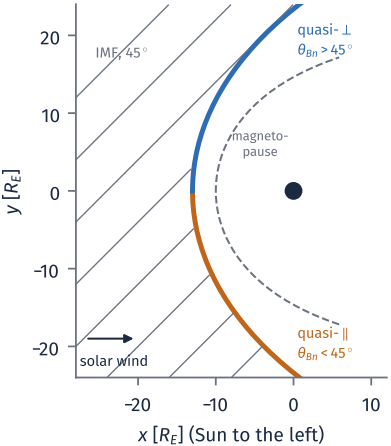
- what replaces viscosity: electric fields in the ramp, ions reflected by them, waves and turbulence (wave–particle interaction, modules 6–8)
- the conservation laws still fix r, p_2, B_2 on scales $\gg c/\omega_{pi}$; they do not fix how the heat is shared: often $T_{i2} \neq T_{e2}$, and the downstream distribution is not Maxwellian

The Earth's bow shock

$c_s = 57$ km/s, $v_A = 49$ km/s: $M_A \approx 8$, $M_f \approx 5$. A fast shock stands in front of the magnetopause ($\approx 10\,R_E$), at $\approx 13\,R_E$.

- quasi-perpendicular, $\theta_{Bn} > 45°$: reflected ions gyrate back into the shock; a sharp ramp with a foot $\sim V/\Omega_i$
- quasi-parallel, $\theta_{Bn} < 45°$: reflected ions escape upstream along $\boldsymbol{B}$, the foreshock, full of waves; the shock is turbulent and broad

Above a critical Mach number ($M_A \approx 2.76$, perpendicular, $\beta \to 0$; Marshall 1955) resistive dissipation cannot make the downstream flow subsonic: ion reflection must take over. Most astrophysical shocks are supercritical.



IMF along the Parker spiral at 45° (module 12a)

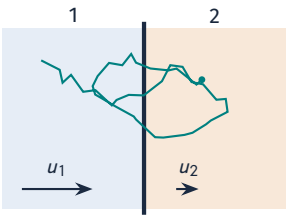
Diffusive shock acceleration

Fast particles at a shock

Test particles with $v \gg u_1$. Magnetic irregularities on both sides scatter them in angle, so they are isotropic in the local fluid frame, and they cross the shock many times.

- shock frame: upstream fluid arrives at u_1, downstream leaves at $u_2 = u_1/r$
- seen from either side, the other fluid approaches at $\Delta u = u_1 - u_2$
- converging scatterers: every crossing is a head-on collision, the closing mirrors of module 4b

We need the two numbers of module 4b: the gain per cycle and the escape probability per cycle.



The energy gain per cycle

A particle of momentum p, isotropic in frame 1, crosses into frame 2, which moves towards it at Δu. Lorentz transformation, first order in $\Delta u/v$, $\mu = \cos\theta$ to the normal:

$$p' = p + \frac{E}{c^2}\,\Delta u\,\mu \quad \Rightarrow \quad \frac{\Delta p}{p} = \frac{\Delta u}{v}\,\mu \qquad (E/pc^2 = 1/v)$$

Which μ? The rate of crossing a plane is $\propto v\mu$: of the particles that cross, a fraction $2\mu\,\mathrm{d}\mu$ has μ in $\mathrm{d}\mu$ $(0 < \mu < 1)$:

$$\left\langle \frac{\Delta p}{p} \right\rangle_{1\to 2} = \frac{\Delta u}{v} \int_0^1 \mu\, 2\mu\,\mathrm{d}\mu = \frac{2}{3}\,\frac{\Delta u}{v}$$

Back from 2 to 1: the same, since frame 1 approaches frame 2 at Δu too. One cycle $1 \to 2 \to 1$:

Key result

$$\xi = \left\langle \frac{\Delta p}{p} \right\rangle_{\text{cycle}} = \frac{4}{3}\,\frac{u_1 - u_2}{v}$$

first order: no losing collisions

The escape probability

- upstream, the flow carries every particle back to the shock: no escape
- downstream, far from the shock the particles are isotropic with density n and are carried away at u_2: loss flux nu_2

Flux of particles crossing the shock from 1 to 2 (isotropic, density n):

$$\Phi_{\rm cross} = n\,\frac{1}{4\pi}\int v\cos\theta\,\mathrm{d}\Omega\Big|_{\cos\theta>0} = n\,\frac{v}{2}\int_0^1 \mu\,\mathrm{d}\mu = \frac{nv}{4}$$

The fraction of the crossing particles that never return:

Key result

$$P = \frac{nu_2}{nv/4} = \frac{4u_2}{v}$$

Small, like ξ: a particle makes $\sim v/4u_2$ cycles on average.

The spectrum depends only on the compression

Module 4b: $N(>p) \propto p^{-P/\xi}$ for $P, \xi \ll 1$. Here

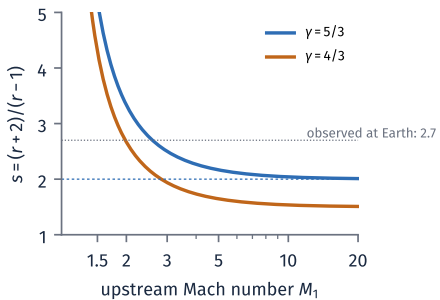
$$\frac{P}{\xi} = \frac{4u_2/v}{\frac{4}{3}(u_1 - u_2)/v} = \frac{3u_2}{u_1 - u_2} = \frac{3}{r-1}$$

The particle speed v cancels, and with it the scattering details. Differentiating, $N(p)\,\mathrm{d}p \propto p^{-s}\mathrm{d}p$:

Key result

$$s = 1 + \frac{3}{r-1} = \frac{r+2}{r-1}, \qquad s = 2 \text{ for } r = 4$$

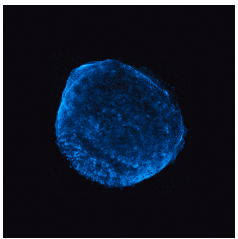
- relativistic, $E = pc$: $N(E) \propto E^{-2}$
- synchrotron $F_\nu \propto \nu^{-(s-1)/2} = \nu^{-0.5}$: radio remnants are close



Supernova remnants and cosmic rays

Are supernova remnants the source?

- **energy**: Galactic cosmic rays need $\sim 10^{41}$ erg/s; 10^{51} erg every 30–50 yr give $\sim 10^{42}$ erg/s: 10% efficiency suffices
- **spectrum**: E^{-2} at the source, steepened to $E^{-2.7}$ by energy-dependent escape
- **maximum energy** (Bohm diffusion, notes): $E_{\max} \approx \frac{3}{20} eBu_1^2 t/c$; at 5000 km/s for 10^3 yr, 4×10^{13} eV if $B = 3\ \mu$G, 10^{15} eV if $100\ \mu$G
- thin X-ray rims: $B \sim 100\ \mu$G at the shock



NASA/CXC/Rutgers/G. Cassam-Chenaï, J. Hughes et al.

Outlook: the cosmic rays amplify the field themselves; their current upstream drives an instability (Bell 2004).

People

Historical note

- Riemann (1860): nonlinear sound waves steepen; Rankine (1870), Hugoniot (1887–89): the jump conditions
- Sedov (1946), Taylor (1950): the point explosion; Taylor obtained the yield of the first atomic bomb from published photographs
- de Hoffmann & Teller (1950): MHD shocks; Marshall (1955): the critical Mach number
- Fermi (1949): cosmic rays accelerated by moving magnetic clouds
- Krymskii (1977), Axford, Leer & Skadron (1977), Bell (1978), Blandford & Ostriker (1978): diffusive shock acceleration and $s = (r+2)/(r-1)$
- Bell (2004): cosmic rays amplify the field that confines them

Check your understanding

1. Why does a compressive wave steepen, and an expansion wave spread out?
2. Why can the jump conditions be derived without knowing the viscosity?
3. Why is the compression of a strong shock bounded, while its pressure jump is not?
4. A supernova remnant is 20 pc across after 10^4 yr, in gas with $n = 1$ cm^{-3}: estimate E.
5. Show that in a perpendicular shock $B_2/B_1 = \rho_2/\rho_1$. Why can a perpendicular shock not be a slow shock?
6. Why is the DSA index independent of the scattering mechanism? What does $r = 2.5$ give?

Exercises in chapter 14 of the notes.

So far, and next

In this module

- steepening: $\mathrm{d}x/\mathrm{d}t = c_0 + \frac{\gamma+1}{2}u$; breaking time t_b
- Rankine–Hugoniot: $r(M_1) \le (\gamma+1)/(\gamma-1) = 4$, $T_2 = \frac{3}{16}\mu m_p V_s^2$, $\Delta s \propto (M_1^2 - 1)^3$
- Sedov–Taylor: $R = 1.15(E t^2/\rho)^{1/5}$
- MHD: $[B_n] = 0$, coplanarity; fast, slow, intermediate; switch-on/off
- collisionless: thickness $\sim c/\omega_{pi}$; quasi-$\perp$/quasi-$\parallel$; $M_c \approx 2.76$
- DSA: $\xi = \frac{4}{3}\Delta u/v$, $P = 4u_2/v$, $s = (r+2)/(r-1) = 2$

Next: module 15. Magnetic reconnection: where flux freezing fails, Sweet–Parker and Petschek, and fast collisionless reconnection.