Plasma Physics and Dynamo Theory • Module 13

Local stability of rotating, magnetised plasmas

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Outline

1. Equilibrium of a rotating, magnetised cylinder
2. Local (WKB) perturbations in cylindrical geometry
3. Toroidal fields: the Tayler instability
4. Red giants: an instability that transports angular momentum
5. Differential rotation: the magnetorotational instability; accretion discs

Reading

Tayler (1973); Acheson (1978)
Spruit (1999, 2002)
Bonanno & Urpin (2008)
Balbus & Hawley (1991, 1998)

Notes: chapter 13

Where we are

- module 12a: the Lorentz force is magnetic pressure plus tension; field lines are frozen in; differential rotation winds a poloidal field into a toroidal one, $\partial_t B_\phi = s(\boldsymbol{B}_p \cdot \nabla)\Omega$
- module 12b: small perturbations of a *uniform* plasma are waves with real ω

Now the background has gradients: a current (B_ϕ varying with s) and a shear (Ω varying with s). Both store free energy. Two questions:

1. what holds such a configuration in equilibrium?
2. does a small perturbation grow? We answer locally: wavelengths short compared with s, so that the background enters only through its local values and gradients.

Equilibrium

A rotating, magnetised cylinder

Cylindrical (s, ϕ, z), everything depends on s only:

$$\boldsymbol{U} = s\,\Omega(s)\,\hat{\boldsymbol{\phi}}, \qquad \boldsymbol{B} = B_\phi(s)\,\hat{\boldsymbol{\phi}} + B_z\,\hat{\boldsymbol{z}} \quad (B_z \text{ uniform}), \qquad \rho \text{ uniform}.$$

$\nabla \times \boldsymbol{B} = \frac{1}{s}\frac{\mathrm{d}(sB_\phi)}{\mathrm{d}s}\hat{\boldsymbol{z}}$, so $(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = -\frac{B_\phi}{s}\frac{\mathrm{d}(sB_\phi)}{\mathrm{d}s}\hat{\boldsymbol{s}}$; $(\boldsymbol{U}\cdot\nabla)\boldsymbol{U} = -s\Omega^2\hat{\boldsymbol{s}}$. Radial momentum balance, with gravity $-g\hat{\boldsymbol{s}}$:

$$-\rho s\Omega^2 = -\frac{\mathrm{d}p}{\mathrm{d}s} - \rho g - \frac{B_\phi}{4\pi s}\frac{\mathrm{d}(sB_\phi)}{\mathrm{d}s} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(p + \frac{B_\phi^2}{8\pi}\right) - \rho g - \frac{B_\phi^2}{4\pi s}$$

$B_\phi^2/4\pi s$: the hoop stress of the curved toroidal lines, pulling towards the axis. Pinch ($\Omega = g = 0$): p against magnetic pressure and hoop stress; thin disc: $s\Omega^2 = GM/s^2$, Keplerian. The balance fixes $p(s)$, not the stability. That is set by two gradients:

$$\alpha \equiv \frac{\mathrm{d}\ln B_\phi}{\mathrm{d}\ln s} \text{ (current)}, \qquad q \equiv -\frac{\mathrm{d}\ln\Omega}{\mathrm{d}\ln s} \text{ (shear; Keplerian } q = \tfrac{3}{2})$$

Local perturbations

Linearising in cylindrical coordinates: the curvature terms

Perturbations $\boldsymbol{u}$, $\boldsymbol{b}$, total pressure $\Pi = p_1 + \boldsymbol{B}\cdot\boldsymbol{b}/4\pi$; ideal, incompressible:

$$\partial_t \boldsymbol{u} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{U} = -\frac{\nabla\Pi}{\rho} + \frac{(\boldsymbol{B}\cdot\nabla)\boldsymbol{b} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{B}}{4\pi\rho}, \qquad \partial_t \boldsymbol{b} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B} + \boldsymbol{U}\times\boldsymbol{b})$$

The unit vectors turn with ϕ: $\partial_\phi \hat{s} = \hat{\phi}$, $\partial_\phi \hat{\phi} = -\hat{s}$. Hence, e.g.,

$$(\boldsymbol{U}\cdot\nabla)\boldsymbol{u} = \Omega\,\partial_\phi \boldsymbol{u} = \Omega\big(\partial_\phi u_s - u_\phi\big)\hat{s} + \Omega\big(\partial_\phi u_\phi + u_s\big)\hat{\phi} + \Omega\,\partial_\phi u_z\,\hat{z}$$

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{U} = u_s \frac{\mathrm{d}(s\Omega)}{\mathrm{d}s}\hat{\phi} - \Omega u_\phi\,\hat{s}$$

Sum: $\Omega\partial_\phi \boldsymbol{u} - 2\Omega u_\phi \hat{s} + (2\Omega + s\Omega')u_s\hat{\phi}$: Coriolis and epicyclic terms. The magnetic terms are built the same way, with B_ϕ/s in place of Ω.

The linearised equations, component by component

Field in velocity units: $\boldsymbol{b} \to \boldsymbol{b}/\sqrt{4\pi\rho}$, $\omega_B \equiv B_\phi/(s\sqrt{4\pi\rho})$, $D_t \equiv \partial_t + \Omega\partial_\phi$, $\mathcal{B} \equiv (B_\phi s^{-1}\partial_\phi + B_z\partial_z)/\sqrt{4\pi\rho}$, $P = \Pi/\rho$:

$$
\begin{aligned}
D_t u_s - 2\Omega u_\phi &= -\partial_s P + \mathcal{B} b_s - 2\omega_B b_\phi & \qquad D_t b_s &= \mathcal{B} u_s \\
D_t u_\phi + (2\Omega + s\Omega')u_s &= -s^{-1}\partial_\phi P + \mathcal{B} b_\phi + (1+\alpha)\,\omega_B b_s & D_t b_\phi &= s\Omega'\, b_s + \mathcal{B} u_\phi - (\alpha-1)\,\omega_B u_s \\
D_t u_z &= -\partial_z P + \mathcal{B} b_z & D_t b_z &= \mathcal{B} u_z
\end{aligned}
$$

- $-2\omega_B b_\phi$: perturbed hoop stress; $(1+\alpha)\omega_B b_s$: tension of the bent background lines
- $s\Omega' b_s$: the shear stretches b_s into b_ϕ, the Ω effect of module 12a
- $-(\alpha-1)\omega_B u_s$: a radial displacement carries B_ϕ into a region where B_ϕ differs

Local (WKB) modes

Wavelengths short compared with s: the coefficients are frozen at their local values and

$$(\boldsymbol{u}, \boldsymbol{b}, P) \propto \exp(\sigma t + i k_s s + i m \phi + i k_z z), \qquad |\boldsymbol{k}| s \gg 1, \quad \boldsymbol{k} = (k_s,\, m/s,\, k_z)$$

- $\partial \to i\boldsymbol{k}$; $D_t \to \tilde{\sigma} = \sigma + im\Omega$ (Doppler shift); $\mathcal{B} \to i\omega_A$, with
  $\omega_A = \dfrac{\boldsymbol{k}\cdot\boldsymbol{B}}{\sqrt{4\pi\rho}} = \dfrac{mB_\phi/s + k_z B_z}{\sqrt{4\pi\rho}}$
- incompressible: $\boldsymbol{k}\cdot\boldsymbol{u} = 0$ (and $\boldsymbol{k}\cdot\boldsymbol{b} = 0$)
- the curvature and gradient terms stay: they are frequencies (Ω, ω_B), not small

A non-zero solution needs a vanishing determinant: the dispersion relation $\sigma(\boldsymbol{k})$. Two cases: **Tayler**: $\Omega = 0$, $B_z = 0$ (current only); **MRI**: $B_\phi = 0$, $m = 0$ (shear only).

Toroidal fields: the Tayler instability

Tayler (1): a toroidal field alone

No rotation, no axial field: $\Omega = 0$, $B_z = 0$. Then $\tilde{\sigma} = \sigma$ and $\omega_A = m\,\omega_B$. The induction equations give $\boldsymbol{b}$ in terms of $\boldsymbol{u}$:

$$b_s = \frac{i\omega_A}{\sigma} u_s, \qquad b_\phi = \frac{i\omega_A u_\phi - (\alpha - 1)\omega_B u_s}{\sigma}, \qquad b_z = \frac{i\omega_A}{\sigma} u_z$$

Insert in the momentum equations and multiply by σ. With $X \equiv \sigma^2 + \omega_A^2$:

$$\begin{aligned}
\left[X - 2(\alpha - 1)\omega_B^2\right] u_s + 2i\omega_A\omega_B\, u_\phi + i\sigma k_s P = 0 \qquad & (a)\\
-2i\omega_A\omega_B\, u_s + X\, u_\phi + i\sigma k_\phi P = 0 \qquad & (b)\\
X\, u_z + i\sigma k_z P = 0 \qquad & (c)\\
k_s u_s + k_\phi u_\phi + k_z u_z = 0 \qquad & (d)
\end{aligned}$$

In (b): $i\omega_A\big[-(\alpha - 1)\big] + i\omega_A(1 + \alpha) = 2i\omega_A$, times ω_B. Without curvature ($\omega_B \to 0$, ω_A fixed), $X = 0$: the Alfvén wave of module 12b.

Tayler (2): the dispersion relation

Take $k_s = 0$; then (d): $k_\phi u_\phi + k_z u_z = 0$. $k_\phi \times (b) + k_z \times (c)$, with (d): $-2i\omega_A\omega_B k_\phi u_s + i\sigma k^2 P = 0, \quad k^2 = k_\phi^2 + k_z^2$. Back in (b):

$$X\, u_\phi = 2i\omega_A\omega_B\Big(1 - \frac{k_\phi^2}{k^2}\Big)u_s = 2i\omega_A\omega_B \frac{k_z^2}{k^2}\, u_s$$

Multiply (a) by X and insert:

$$\big[X - 2(\alpha - 1)\omega_B^2\big] X\, u_s + 2i\omega_A\omega_B \cdot 2i\omega_A\omega_B \frac{k_z^2}{k^2}\, u_s = 0$$

Key result

$$X^2 + 2(1-\alpha)\,\omega_B^2\, X - 4\omega_A^2\omega_B^2\, \frac{k_z^2}{k^2} = 0, \qquad X = \sigma^2 + \omega_A^2$$

$k_s \neq 0$ only dilutes the drive (notes): the most unstable modes have $k_s = 0$.

Tayler (3): sausage and kink

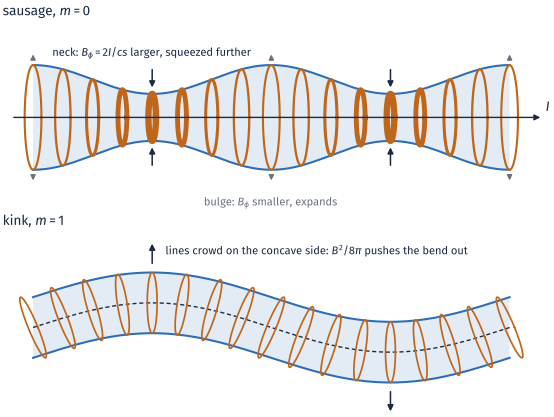
Short vertical wavelengths, $k_z \gg m/s$:
$k_z^2/k^2 \to 1$. $m = 0$: $\omega_A = 0$,
$X = \sigma^2 = 2(\alpha - 1)\,\omega_B^2$. $m \geq 1$:
$\omega_A = m\omega_B$, positive root

$$\frac{\sigma^2}{\omega_B^2} = \sqrt{(1-\alpha)^2 + 4m^2} - (1-\alpha) - m^2,$$

positive iff $4 > m^2 + 2(1-\alpha)$.

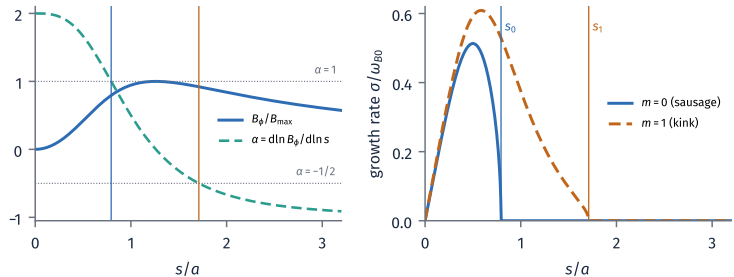
Key result

sausage, $m = 0$: unstable if $\alpha > 1$
$m \geq 1$: unstable if $\alpha > \frac{m^2}{2} - 1$
kink, $m = 1$: unstable if $\alpha > -\frac{1}{2}$



Tayler (4): which profiles are unstable, and where

The criteria hold radius by radius. Example, a hollow current in $s \lesssim a$: $B_\phi = B_0 x^2/(1+x^3)$, $\alpha = (2-x^3)/(1+x^3)$, $x = s/a$. Sausage where $\alpha > 1$: $s < s_0 = 2^{-1/3} a$; kink where $\alpha > -\frac{1}{2}$: $s < s_1 = 5^{1/3} a$; outside, $B_\phi \propto 1/s$ (no current, $\alpha = -1$): stable.



Near the axis a regular field has $B_\phi \propto s$, $\alpha = 1$: every toroidal field is kink unstable there, with $\sigma \sim \omega_B$, an Alfvén time (Tayler 1973).

Red giants (1): the cores rotate too slowly

After the main sequence the core contracts and the envelope expands. If each shell kept its angular momentum, the core would spin up by orders of magnitude.

- **mixed modes** (Kepler): gravity waves in the core coupled to pressure waves in the envelope. Their rotational splitting measures the rotation of the core (Beck et al. 2012; Deheuvels et al. 2012)
- the core rotates only about ten times faster than the envelope, and spins *down* as the star ascends the giant branch (Mosser et al. 2012)
- models with hydrodynamic transport only: cores one to two orders of magnitude too fast

Magnetic fields in red-giant cores

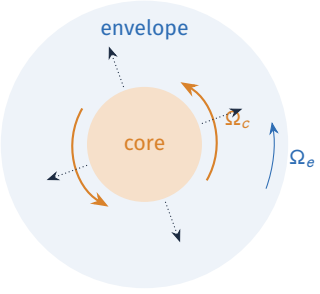
Suppressed dipole modes in some red giants point to strong core fields (Fuller et al. 2015); asymmetries of the mixed-mode multiplets have measured core fields of 30–100 kG (Li et al. 2022). The field is there; can it remove the angular momentum?

Red giants (2): an instability that transports angular momentum

The Tayler–Spruit loop (Spruit 2002):

1. the core spins faster than the envelope: the shear winds a toroidal field (Ω effect, module 12a)
2. B_ϕ grows until it is Tayler unstable (kink, $m = 1$)
3. the unstable motions create B_s, wound up again
4. the stress $-B_s B_\phi / 4\pi$ carries angular momentum outwards

Original estimate: cores still too fast (Cantiello et al. 2014); revised saturation: close to the observations (Fuller et al. 2019).



angular momentum out:
$\Omega_c \simeq 10\,\Omega_e$, not 10^3

Differential rotation: the MRI

MRI (1): an axial field in a shear flow

$B_\phi = 0$, uniform B_z, $\Omega(s)$ with $s\Omega' = -q\Omega$; axisymmetric modes ($m = 0$), so $\tilde{\sigma} = \sigma$, $\omega_A = k_z v_A$. From the component equations:

$$\begin{aligned} \sigma u_s - 2\Omega u_\phi &= -ik_s P + i\omega_A b_s, & \sigma b_s &= i\omega_A u_s \\ \sigma u_\phi + (2-q)\Omega\, u_s &= i\omega_A b_\phi, & \sigma b_\phi &= -q\Omega\, b_s + i\omega_A u_\phi \\ \sigma u_z &= -ik_z P + i\omega_A b_z, & \sigma b_z &= i\omega_A u_z, \qquad k_s u_s + k_z u_z = 0 \end{aligned}$$

Eliminate $\boldsymbol{b}$, multiply by σ ($X = \sigma^2 + \omega_A^2$); the z equation and $\boldsymbol{k}\cdot\boldsymbol{u} = 0$ give $i\sigma P = X k_s u_s / k_z^2$:

$$\frac{k^2}{k_z^2} X\, u_s = 2\Omega\sigma\, u_\phi \qquad (A)$$

$$X\, u_\phi + \left[(2-q)\Omega\sigma - q\Omega\frac{\omega_A^2}{\sigma}\right] u_s = 0 \qquad (B)$$

MRI (2): Rayleigh's criterion, and why a field breaks it

$2\Omega\sigma \times (B)$, with (A):

Key result

$$\frac{k^2}{k_z^2}\left(\sigma^2 + \omega_A^2\right)^2 + \kappa^2\sigma^2 - 2q\Omega^2\omega_A^2 = 0, \qquad \kappa^2 = 2(2-q)\Omega^2 = \frac{1}{s^3}\frac{\mathrm{d}(s^2\Omega)^2}{\mathrm{d}s}$$

No field: $\sigma^2 = -\kappa^2 k_z^2/k^2$, stable iff $\ell = s^2\Omega$ grows outwards (Rayleigh 1917): a ring pushed out keeps its ℓ, has too little centrifugal force there, and falls back. Keplerian, $\ell \propto s^{1/2}$: stable. **With a field**: quadratic in σ^2, positive leading coefficient, value $\omega_A^2(k^2 v_A^2 - 2q\Omega^2)$ at $\sigma^2 = 0$: a positive root iff $k^2 v_A^2 < 2q\Omega^2 = -\mathrm{d}\Omega^2/\mathrm{d}\ln s$. The field ties the ring to its neighbours, so ℓ is no longer conserved: the criterion is on Ω. Any weak field destabilises a Keplerian disc (Velikhov 1959; Chandrasekhar 1960; Balbus & Hawley 1991).

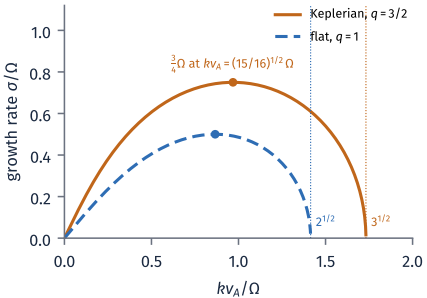
MRI (3): how fast, and a spring between two orbits

Maximise over ω_A^2 ($k_s = 0$): σ^2 is stationary where $\omega_A^2 = q\Omega^2 - \sigma^2$; then $4\Omega^2\sigma^2 = q^2\Omega^4$.

Key result

$\sigma_{\max} = \frac{q}{2}\Omega$ at $k_z^2 v_A^2 = q(1 - \frac{q}{4})\Omega^2$. Keplerian: $\frac{3}{4}\Omega$, a factor $e^{3\pi/2} \approx 110$ per orbit, whatever B.

Two elements on one field line, joined by the tension like a spring: the inner one orbits faster, the spring pulls it back, it loses angular momentum and sinks; the outer one is pulled forward and rises. Runaway, unless the spring is stiff, $k^2 v_A^2 > 2q\Omega^2$.



Accretion discs: angular momentum must go out

A ring at s has $\ell = s^2\Omega = \sqrt{GMs}$. To move in, it must give angular momentum to the gas outside: accretion *is* outward transport of angular momentum.

- molecular viscosity is far too small for the observed accretion rates (Pringle 1981)
- on the growing MRI mode, from (A) and the induction equation, $\dfrac{u_\phi}{u_s} = \dfrac{X}{2\Omega\sigma} > 0$, $\dfrac{b_\phi}{b_s} = -\dfrac{2q\Omega^2 - X}{2\Omega\sigma} < 0$: the stress $\rho\,(u_s u_\phi - b_s b_\phi) > 0$ carries angular momentum outwards
- the instability saturates in MHD turbulence; transport is then turbulent, $\nu_t = \alpha c_s H$ (Shakura & Sunyaev 1973), $\alpha \sim 10^{-2}$–10^{-1} in simulations, mostly Maxwell stress. Turbulent coefficients are not molecular ones: module 17
- where the gas is too weakly ionised the MRI dies: dead zones of protoplanetary discs (Gammie 1996)

Synthesis

Two instabilities from one local analysis

	Tayler	**MRI**
free energy	current, $B_\phi(s)$	shear, $\Omega(s)$
criterion	$m = 0$: $\alpha > 1$; $m = 1$: $\alpha > -\frac{1}{2}$	$\mathrm{d}\Omega^2/\mathrm{d}s < 0$, $k^2 v_A^2 < 2q\Omega^2$
growth rate	$\sim \omega_B$	$\frac{q}{2}\Omega$
where, what	red-giant cores: core–envelope coupling	accretion discs: turbulent transport

Topics for further study (possible exam projects):

- Tayler with a weak B_z, threshold $\alpha > -1$ (Bonanno & Urpin 2008); saturation and helicity (Bonanno et al. 2012)
- B_ϕ, B_z and Ω' together: helical MRI, shearing waves; red giants and MRI saturation

People

Historical note

- Rayleigh (1917): centrifugal stability, angular momentum must grow outwards
- Velikhov (1959), Chandrasekhar (1960): rotating Couette flow destabilised by an axial field
- Tayler (1973): toroidal fields in stars are unstable to $m = 1$; Acheson (1978): the local theory
- Balbus & Hawley (1991): the same instability makes accretion discs turbulent
- Spruit (2002): the Tayler–Spruit dynamo in stellar radiative zones
- Beck et al., Mosser et al. (2012): the rotation of red-giant cores from Kepler asteroseismology

Check your understanding

1. Where do the terms $-2\Omega u_\phi$ and $(2\Omega + s\Omega')u_s$ come from? Why are there no such terms in Cartesian coordinates?
2. Show that a displaced ring of toroidal field keeps B_ϕ/s. Why is $\alpha > 1$ unstable for $m = 0$?
3. For $B_\phi \propto s^\alpha$, which m are unstable if $\alpha = 2$?
4. Why is every toroidal field unstable near the axis, whatever its profile further out?
5. A Keplerian disc is Rayleigh-stable. Why does an arbitrarily weak field change that?
6. Why must the core of a red giant lose angular momentum, and how can a Tayler-unstable field help?

Exercises in chapter 13 of the notes.

So far, and next

In this module

- cylindrical equilibrium: pressure, hoop stress $B_\phi^2/4\pi s$, centrifugal force, gravity
- local modes: Coriolis, epicyclic and curvature terms
- Tayler: sausage for $\alpha > 1$, kink for $\alpha > -\frac{1}{2}$; critical radii; red-giant cores
- MRI: unstable if $\mathrm{d}\Omega^2/\mathrm{d}s < 0$, $\sigma_{\max} = \frac{3}{4}\Omega$; outward angular momentum transport

Next: module 14. Shocks: when perturbations are not small, MHD waves steepen; the jump conditions, and particle acceleration at shocks.