Plasma Physics and Dynamo Theory • Module 12b

Linear MHD waves and their dissipation

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Outline

1. Linearised ideal MHD about a uniform magnetised fluid
2. The eigenvalue problem for the velocity
3. The Alfvén wave
4. The fast and slow magnetosonic waves: dispersion, limits, polarisation
5. Phase and group velocity: Friedrichs diagrams
6. Dissipation: resistive and viscous damping, phase mixing

Reading

Goedbloed & Poedts, ch. 5
Kulsrud, ch. 4

Notes: chapter 12

Where we are

Module 12a: the Lorentz force is a magnetic pressure $B^2/8\pi$ across the field and a tension $B^2/4\pi$ along it; the field lines are frozen into the fluid.

- a stretched string carries transverse waves: (speed)^2 = tension / mass per unit length
- a gas carries sound because it resists compression
- a magnetised fluid has tension, magnetic pressure and gas pressure, all at once

Today's question: what are the small oscillations of a uniform magnetised fluid? How fast do they travel, in which direction does their energy go, what do they compress, and what damps them?

Method: linearise ideal MHD about rest, look for plane waves, and reduce everything to an eigenvalue problem for the velocity.

Linear MHD waves

Linearisation about a uniform state

Equilibrium: ρ_0, p_0 uniform, $\boldsymbol{B}_0 = B_0\hat{z}$ uniform, $\boldsymbol{U}_0 = 0$. It is one: no gradients, and $\boldsymbol{j}_0 = (c/4\pi)\nabla \times \boldsymbol{B}_0 = 0$.

Perturb: $\rho = \rho_0 + \rho_1$, $p = p_0 + p_1$, $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}$, $\boldsymbol{U} = \boldsymbol{u}$, and keep first-order terms.

- $\boldsymbol{u} \cdot \nabla$ acting on a perturbation is second order: $\mathrm{d}/\mathrm{d}t \to \partial_t$
- $\boldsymbol{j} \times \boldsymbol{B} = \boldsymbol{j}_1 \times \boldsymbol{B}_0 + O(2)$, since $\boldsymbol{j}_0 = 0$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{u} = 0, \qquad \rho_0 \frac{\partial \boldsymbol{u}}{\partial t} = -\nabla p_1 + \frac{(\nabla \times \boldsymbol{b}) \times \boldsymbol{B}_0}{4\pi},$$

$$\frac{\partial}{\partial t}\left(p_1 - c_s^2 \rho_1\right) = 0, \qquad \frac{\partial \boldsymbol{b}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}_0), \qquad c_s^2 = \frac{\gamma p_0}{\rho_0}$$

The adiabatic law, $\delta(p\rho^{-\gamma}) = 0$, gives $p_1/p_0 = \gamma\rho_1/\rho_0$: so $p_1 = c_s^2\rho_1$ for a perturbation that starts from rest. Eight equations, constant coefficients.

Plane waves: everything in terms of $\boldsymbol{u}$

Constant coefficients: the equations are invariant under translations, so Fourier modes evolve independently and any perturbation is a superposition of $\exp[i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)]$: $\partial_t \to -i\omega$, $\nabla \to i\boldsymbol{k}$.

The equilibrium is also invariant under rotations about $\boldsymbol{B}_0$: choose x along the part of $\boldsymbol{k}$ across $\boldsymbol{B}_0$,

$$\boldsymbol{k} = (k_\perp, 0, k_\parallel) = k(\sin\theta, 0, \cos\theta), \qquad \theta = \text{angle between } \boldsymbol{k} \text{ and } \boldsymbol{B}_0$$

Continuity, adiabatic law and induction give ρ_1, p_1, $\boldsymbol{b}$ directly in terms of $\boldsymbol{u}$. With $\boldsymbol{k}\times(\boldsymbol{u}\times\boldsymbol{B}_0) = \boldsymbol{u}\,(\boldsymbol{k}\cdot\boldsymbol{B}_0) - \boldsymbol{B}_0(\boldsymbol{k}\cdot\boldsymbol{u})$:

$$\rho_1 = \rho_0\frac{\boldsymbol{k}\cdot\boldsymbol{u}}{\omega}, \qquad p_1 = c_s^2\rho_0\frac{\boldsymbol{k}\cdot\boldsymbol{u}}{\omega}, \qquad \boldsymbol{b} = -\frac{B_0}{\omega}\left[k_\parallel\boldsymbol{u} - (\boldsymbol{k}\cdot\boldsymbol{u})\,\hat{\boldsymbol{z}}\right]$$

Only the momentum equation is left: three equations for the three components of $\boldsymbol{u}$. An eigenvalue problem.

The restoring force

Lorentz force, with $(\boldsymbol{k}\times\boldsymbol{b})\times\boldsymbol{B}_0 = (\boldsymbol{k}\cdot\boldsymbol{B}_0)\,\boldsymbol{b} - (\boldsymbol{b}\cdot\boldsymbol{B}_0)\,\boldsymbol{k}$:

$$\frac{(i\boldsymbol{k}\times\boldsymbol{b})\times\boldsymbol{B}_0}{4\pi} = \frac{iB_0}{4\pi}\Big[\underbrace{k_\parallel\boldsymbol{b}}_{\text{tension}} - \underbrace{b_z\,\boldsymbol{k}}_{\text{magnetic pressure}}\Big]$$

the bending of the lines, and the gradient of $B_0 b_z/4\pi$, the perturbation of $B^2/8\pi$.
Momentum, $-i\omega\rho_0\boldsymbol{u} = -i\boldsymbol{k}\,p_1 + (\ldots)$: insert p_1 and $\boldsymbol{b}$, using $b_z = (B_0/\omega)\big[\boldsymbol{k}\cdot\boldsymbol{u} - k_\parallel u_z\big]$, and multiply by $i\omega/\rho_0$:

$$\omega^2\boldsymbol{u} = c_s^2\,\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{u}) + v_A^2\Big\{k_\parallel\big[k_\parallel\boldsymbol{u} - (\boldsymbol{k}\cdot\boldsymbol{u})\hat{\boldsymbol{z}}\big] + \boldsymbol{k}\big[\boldsymbol{k}\cdot\boldsymbol{u} - k_\parallel u_z\big]\Big\}, \qquad v_A^2 = \frac{B_0^2}{4\pi\rho_0}$$

The coefficient of the magnetic terms, $B_0^2/4\pi\rho_0$, is a speed squared: the Alfvén speed v_A, tension per unit mass density, as for a string.

The eigenvalue problem in components

Write out $\boldsymbol{k} \cdot \boldsymbol{u} = k_\perp u_x + k_\parallel u_z$ in each component:

$$
\begin{aligned}
x: &\quad c_s^2 k_\perp (k_\perp u_x + k_\parallel u_z) + v_A^2 \left[k_\parallel^2 u_x + k_\perp^2 u_x \right] \\
y: &\quad v_A^2 k_\parallel^2 u_y \\
z: &\quad c_s^2 k_\parallel (k_\perp u_x + k_\parallel u_z) + v_A^2 \left[k_\parallel^2 u_z - k_\parallel (\boldsymbol{k} \cdot \boldsymbol{u}) + k_\parallel (\boldsymbol{k} \cdot \boldsymbol{u}) - k_\parallel^2 u_z \right] = c_s^2 k_\parallel (\boldsymbol{k} \cdot \boldsymbol{u})
\end{aligned}
$$

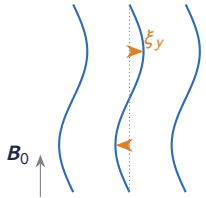
$$
\omega^2 \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} k^2 v_A^2 + k_\perp^2 c_s^2 & 0 & k_\perp k_\parallel c_s^2 \\ 0 & k_\parallel^2 v_A^2 & 0 \\ k_\perp k_\parallel c_s^2 & 0 & k_\parallel^2 c_s^2 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}
$$

- z: no magnetic force along $\boldsymbol{B}_0$; the Lorentz force is perpendicular to $\boldsymbol{B}$ (module 12a)
- y decouples from x, z: the problem is symmetric under $y \to -y$
- the matrix is real and symmetric: ω^2 is real, the three eigenvectors are orthogonal

The Alfvén wave

The y row alone: $\omega^2 = k_\parallel^2 v_A^2$, with $\boldsymbol{u} = u_y \hat{\boldsymbol{y}}$, perpendicular to both $\boldsymbol{k}$ and $\boldsymbol{B}_0$.

- $\boldsymbol{k} \cdot \boldsymbol{u} = 0$: $\rho_1 = p_1 = 0$, incompressible
- $\boldsymbol{b} = -(B_0 k_\parallel/\omega)\, u_y \hat{\boldsymbol{y}}$, so $b_z = 0$: $|\boldsymbol{B}|$ unchanged to first order, no magnetic pressure. The restoring force is the tension alone
- for $\omega = k_\parallel v_A > 0$ (along $+\boldsymbol{B}_0$): $\boldsymbol{b}/(4\pi\rho_0)^{1/2} = -\boldsymbol{u}$, the Walén relation ($+\boldsymbol{u}$ along $-\boldsymbol{B}_0$)



lines displaced, y–z plane

Key result

$\omega = k_\parallel v_A = k v_A \cos\theta$, $v_A = B_0/(4\pi\rho_0)^{1/2}$: a transverse, incompressible wave of the field lines, driven by their tension (Alfvén 1942).

The Alfvén wave: where the energy goes

Group velocity. $\omega = v_A\, \boldsymbol{k}\cdot\hat{z}$ is linear in $\boldsymbol{k}$:

$$\boldsymbol{v}_g = \frac{\partial\omega}{\partial\boldsymbol{k}} = v_A\,\hat{z} \qquad \text{for every } \boldsymbol{k}$$

The energy travels along $\boldsymbol{B}_0$ at v_A, whatever the direction of $\boldsymbol{k}$: each field line is a string that carries its own signal, without spreading across.

Equipartition. With $|\boldsymbol{b}| = (B_0/v_A)|\boldsymbol{u}| = (4\pi\rho_0)^{1/2}|\boldsymbol{u}|$:

$$\frac{|\boldsymbol{b}|^2}{8\pi} = \frac{4\pi\rho_0|\boldsymbol{u}|^2}{8\pi} = \frac{\rho_0|\boldsymbol{u}|^2}{2}$$

kinetic and magnetic perturbation energies are equal at every point and every instant (the cross term $\boldsymbol{B}_0\cdot\boldsymbol{b}/4\pi$ vanishes because $b_z = 0$).

Module 11. The cold-plasma R and L waves at $\omega \ll \Omega_i$ gave $\omega = kv_A/(1+v_A^2/c^2)^{1/2}$: the same wave, seen from the particles. MHD holds for $\omega \ll \Omega_i$, $k\rho_i \ll 1$.

Fast and slow waves: the dispersion relation

The x, z rows: a nonzero (u_x, u_z) requires

$$\det \begin{pmatrix} k^2 v_A^2 + k_\perp^2 c_s^2 - \omega^2 & k_\perp k_\parallel c_s^2 \\ k_\perp k_\parallel c_s^2 & k_\parallel^2 c_s^2 - \omega^2 \end{pmatrix} = 0$$

$$\omega^4 - \omega^2 \Big[k^2 v_A^2 + \underbrace{(k_\perp^2 + k_\parallel^2)}_{k^2} c_s^2 \Big] + k^2 v_A^2 k_\parallel^2 c_s^2 + \underbrace{k_\perp^2 k_\parallel^2 c_s^4 - k_\perp^2 k_\parallel^2 c_s^4}_{0} = 0$$

A quadratic in ω^2, with $k_\parallel = k \cos\theta$:

Key result

$$\frac{\omega^2}{k^2} = \frac{1}{2} \left[v_A^2 + c_s^2 \pm \left((v_A^2 + c_s^2)^2 - 4 v_A^2 c_s^2 \cos^2\theta \right)^{1/2} \right]$$

+: fast, −: slow magnetosonic wave. Both compress ($\boldsymbol{k} \cdot \boldsymbol{u} \neq 0$). Symmetric in $v_A \leftrightarrow c_s$.

How the three branches are ordered

Let $f(X) = X^2 - X k^2 (v_A^2 + c_s^2) + k^2 k_\parallel^2 v_A^2 c_s^2$, with roots $X = \omega_s^2 \le \omega_f^2$. $f(X) \le 0$ means that X lies between the roots.

$$f(k_\parallel^2 c_s^2) = k_\parallel^2 c_s^4 \,(k_\parallel^2 - k^2) = -k_\perp^2 k_\parallel^2 c_s^4 \le 0$$
$$f(k_\parallel^2 v_A^2) = k_\parallel^2 v_A^4 \,(k_\parallel^2 - k^2) \le 0$$
$$f(k^2 v_A^2) = -k^2 k_\perp^2 v_A^2 c_s^2 \le 0, \qquad f(k^2 c_s^2) = -k^2 k_\perp^2 v_A^2 c_s^2 \le 0$$

Key result

$$\omega_s^2 \le k_\parallel^2 \min(v_A^2, c_s^2), \qquad \omega_s \le \omega_A = k_\parallel v_A \le \omega_f, \qquad \omega_f^2 \ge k^2 \max(v_A^2, c_s^2)$$

The names are earned: in every direction slow $\le$ Alfvén $\le$ fast.

Limits: along and across the field

$\theta = 0$ ($k_\perp = 0$): the quadratic factorises, $(\omega^2 - k^2 v_A^2)(\omega^2 - k^2 c_s^2) = 0$. The matrix is diagonal:

- u_z: $\omega = k c_s$, a sound wave along $\boldsymbol{B}_0$; $\boldsymbol{b} = 0$, since $\boldsymbol{u} \parallel \boldsymbol{B}_0$ does not bend the lines
- u_x: $\omega = k v_A$, a transverse wave, degenerate with the Alfvén wave (u_x and u_y equivalent)

fast $= \max(v_A, c_s)$, slow $= \min(v_A, c_s)$: which one is "the sound wave" depends on β.

$\theta = \pi/2$ ($k_\parallel = 0$): $\omega^2[\omega^2 - k^2(v_A^2 + c_s^2)] = 0$.

- fast: $\omega = k(v_A^2 + c_s^2)^{1/2}$, $\boldsymbol{u} = u_x \hat{\boldsymbol{x}} \parallel \boldsymbol{k}$; it compresses gas and field together, $b_z/B_0 = \rho_1/\rho_0$, and both pressures push: $c_s^2 + v_A^2$
- slow and Alfvén: $\omega = 0$; with $k_\parallel = 0$ the lines are not bent and there is no tension

Polarisation of the magnetosonic waves

$u_y = b_y = 0$, $\boldsymbol{u}$ in the $(\boldsymbol{k}, \boldsymbol{B}_0)$ plane. Induction: $b_x = -(B_0/\omega) k_\parallel u_x$, $b_z = (B_0/\omega) k_\perp u_x$, so $\boldsymbol{k} \cdot \boldsymbol{b} = 0$.

Gas against magnetic pressure. The z row gives $u_z = k_\perp k_\parallel c_s^2 u_x / (\omega^2 - k_\parallel^2 c_s^2)$, so

$$\boldsymbol{k} \cdot \boldsymbol{u} = k_\perp u_x \left[1 + \frac{k_\parallel^2 c_s^2}{\omega^2 - k_\parallel^2 c_s^2} \right] = \frac{k_\perp u_x \, \omega^2}{\omega^2 - k_\parallel^2 c_s^2}, \qquad \frac{p_1}{B_0 b_z / 4\pi} = \frac{c_s^2}{v_A^2} \frac{\omega^2}{\omega^2 - k_\parallel^2 c_s^2}$$

By the ordering, $\omega_f^2 \geq k_\parallel^2 c_s^2 \geq \omega_s^2$:

Key result

fast: gas and magnetic pressure perturbations in phase; they add.
slow: in antiphase; they partly cancel, and the wave is slow.

Alfvén: $\boldsymbol{u}$, $\boldsymbol{b}$ along $\hat{y}$, nothing compressed. Fast and slow: $\boldsymbol{u}$, $\boldsymbol{b}$ in the $(\boldsymbol{k}, \boldsymbol{B}_0)$ plane; ρ_1, p_1, $|\boldsymbol{B}|$ perturbed.

Low and high β

$\beta = 8\pi p_0/B_0^2 = (2/\gamma)\, c_s^2/v_A^2$. Expand the square root for $c_s \ll v_A$ or $c_s \gg v_A$:
$\left[(v_A^2 + c_s^2)^2 - 4 v_A^2 c_s^2 \cos^2\theta\right]^{1/2} \simeq v_A^2 + c_s^2 - 2 v_A^2 c_s^2 \cos^2\theta/(v_A^2 + c_s^2)$.

$\beta \ll 1$ (corona, magnetospheres)

- fast: $\omega^2 \simeq k^2(v_A^2 + c_s^2 \sin^2\theta) \simeq k^2 v_A^2$, isotropic: the field's own compressional wave
- slow: $\omega \simeq k_\parallel c_s$, sound guided along $\boldsymbol{B}_0$: the lines act as rigid pipes

$\beta \gg 1$ (stellar interiors)

- fast: $\omega \simeq k c_s$, ordinary sound, slightly stiffened by the field
- slow: $\omega \simeq k_\parallel v_A$; from the polarisation, $\boldsymbol{k} \cdot \boldsymbol{u} \to 0$: nearly incompressible, an Alfvén-like wave polarised in the $(\boldsymbol{k}, \boldsymbol{B}_0)$ plane

Corona ($n = 10^9$ cm^{-3}, $T = 10^6$ K, $B = 10$ G): $v_A \approx 690$ km/s, $c_s \approx 170$ km/s, $\beta \approx 0.07$.

Phase and group velocity

All three branches have $\omega = kV(\theta)$: no dispersion in $|\boldsymbol{k}|$, only anisotropy. Gradient in polar coordinates in the $(k_\perp, k_\parallel)$ plane, $\nabla_k = \hat{\boldsymbol{k}}\,\partial_k + \hat{\boldsymbol{\theta}}\,k^{-1}\,\partial_\theta$:

$$\boldsymbol{v}_g = \frac{\partial \omega}{\partial \boldsymbol{k}} = V(\theta)\,\hat{\boldsymbol{k}} + \frac{\mathrm{d}V}{\mathrm{d}\theta}\,\hat{\boldsymbol{\theta}}$$

the phase velocity plus a component across $\boldsymbol{k}$, wherever V depends on direction.

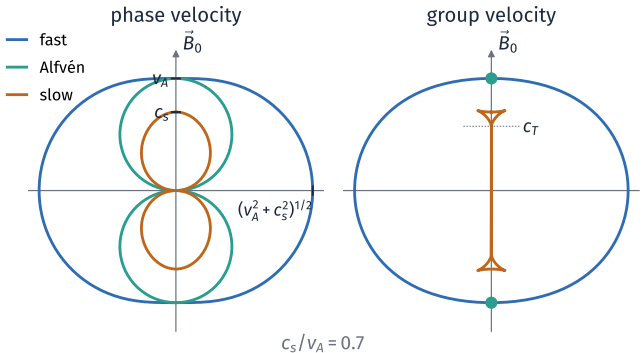
- **phase** (Friedrichs) diagram: $V(\theta)\hat{\boldsymbol{k}}$ as θ goes round; plane fronts after unit time
- **group** diagram: $\boldsymbol{v}_g(\theta)$; the front of a point disturbance after unit time

Slow mode near $\theta = \pi/2$: expand the square root in $\cos^2\theta$,

$$V_s^2 \simeq \frac{v_A^2 c_s^2}{v_A^2 + c_s^2}\cos^2\theta \equiv c_T^2 \cos^2\theta$$

like the Alfvén wave, with the tube speed c_T: there $\boldsymbol{v}_g \to \pm c_T \hat{\boldsymbol{z}}$.

Friedrichs diagrams for $c_s/v_A = 0.7$



- Alfvén: $V = v_A|\cos\theta|$, two circles; group velocity: two points, $\pm v_A\hat{z}$
- fast: nearly isotropic, $\max(v_A, c_s)$ along $\boldsymbol{B}_0$, $(v_A^2 + c_s^2)^{1/2}$ across
- slow: energy only in a narrow cone around $\boldsymbol{B}_0$, between c_T and $\min(v_A, c_s)$, with cusps where the front folds back

Alfvén waves in the corona and in the solar wind

- $v_A = B/(4\pi n m_p)^{1/2}$: ≈ 700 km/s in the corona (10 G, 10^9 cm^{-3}), ≈ 50 km/s at 1 au (5 nT, 5 cm^{-3})
- Mariner 5 (1967): $\delta \boldsymbol{U}$ and $\delta \boldsymbol{B}$ correlated as in the Walén relation, with the sign of waves propagating away from the Sun (Belcher & Davis 1971); now seen by every mission, Parker Solar Probe included
- energy flux (equipartition: energy density $\rho \langle \delta U^2 \rangle$, carried at v_A): $F = \rho \langle \delta U^2 \rangle \, v_A \approx 10^6$ erg cm^{-2} s^{-1} for $\delta U = 30$ km/s, above the losses of the quiet corona, $\sim 3 \times 10^5$ erg cm^{-2} s^{-1}

Enough to heat the corona. But the waves are incompressible and run along the field: how do they give their energy to the plasma?

Dissipation

Resistive and viscous damping

Add viscosity and resistivity. For the Alfvén wave $\nabla \cdot \boldsymbol{u} = 0$, $\boldsymbol{b} \perp \boldsymbol{B}_0$:

$$\frac{\partial \boldsymbol{u}}{\partial t} = \frac{B_0}{4\pi\rho_0}\frac{\partial \boldsymbol{b}}{\partial z} + \nu\nabla^2\boldsymbol{u}, \qquad \frac{\partial \boldsymbol{b}}{\partial t} = B_0\frac{\partial \boldsymbol{u}}{\partial z} + \eta\nabla^2\boldsymbol{b}$$

Plane wave: $(-i\omega + \nu k^2)\boldsymbol{u} = ik_\parallel(B_0/4\pi\rho_0)\boldsymbol{b}$, $(-i\omega + \eta k^2)\boldsymbol{b} = ik_\parallel B_0\boldsymbol{u}$. Eliminate $\boldsymbol{b}$: $(\omega + i\nu k^2)(\omega + i\eta k^2) = k_\parallel^2 v_A^2$:

$$\omega = -\frac{i(\nu+\eta)k^2}{2} \pm \left[k_\parallel^2 v_A^2 - \frac{(\nu-\eta)^2 k^4}{4}\right]^{1/2}$$

Key result

Weak dissipation, $(\nu+\eta)k^2 \ll k_\parallel v_A$: $\omega \simeq \pm k_\parallel v_A - i\gamma$, $\gamma = \dfrac{(\nu+\eta)k^2}{2}$

ν drains the kinetic half of the energy, η the magnetic half. k^2, not $k_\parallel^2$.

How weak is it?

Corona ($T = 10^6$ K, $n = 10^9$ cm^{-3}, $B = 10$ G): Spitzer $\eta \approx 10^4$ cm^2/s (module 2b); the ion viscosity across $\boldsymbol{B}$, $\nu_\perp \approx 0.3\,\rho_i^2/\tau_i \approx 3 \times 10^3$ cm^2/s (module 10): comparable, so use η.

- wavelength 10^4 km: $\tau = 2/(\eta k^2) \approx 5 \times 10^{12}$ s $\approx 1.6 \times 10^5$ yr
- crossing time of a loop, 10^5 km at 700 km/s: ≈ 140 s
- to damp in 100 s, $\lambda = 2\pi(\eta t/2)^{1/2} \approx 40$ m

Key result

At $\lambda = 10^4$ km the wave loses $2\pi\gamma/\omega \approx 3 \times 10^{-12}$ of its amplitude per period. Dissipation needs a process that carries the energy to small scales.

Two candidates: phase mixing (next) and the turbulent cascade.

Phase mixing (1): every field line at its own frequency

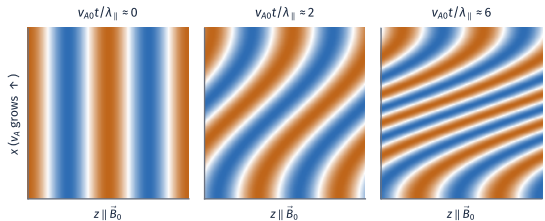
Now $\boldsymbol{B}_0 = B_0\hat{z}$ with $\rho_0(x)$, so $v_A = v_A(x)$ (balance by $p_0 + B_0^2/8\pi$ uniform). Wave $u_y(x,z,t)$, $b_y(x,z,t)$, independent of y: $\nabla\cdot\boldsymbol{u} = 0$, $b_z = 0$, and

$$\rho_0(x)\frac{\partial u_y}{\partial t} = \frac{B_0}{4\pi}\frac{\partial b_y}{\partial z}, \qquad \frac{\partial b_y}{\partial t} = B_0\frac{\partial u_y}{\partial z}$$

No ∂_x: each surface $x = \text{const}$ oscillates alone, at $\omega_A(x) = k_\parallel v_A(x)$. Launched in phase, $u_y = A\cos[k_\parallel z - \omega_A(x)t]$:

$$k_x = \frac{\partial}{\partial x}(\text{phase}) = -k_\parallel v_A'(x)\, t$$

Neighbouring lines drift out of phase: $|k_x|$ grows linearly in time.



Phase mixing (2): the enhanced damping

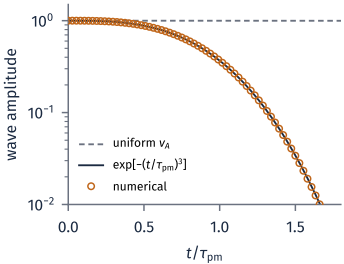
Add ν, η: locally a plane wave with $k^2(t) = k_\perp^2 + k_\parallel^2 v_A'^2 t^2$, damped at $\gamma(t) = (\nu+\eta)k^2(t)/2$. Integrate $\mathrm{d}\ln A/\mathrm{d}t = -\gamma(t)$:

$$A = A_0 \exp\left[-\frac{(\nu+\eta)k_\perp^2 t}{2} - \frac{(\nu+\eta)k_\parallel^2 v_A'^2 t^3}{6} \right]$$

For $v_A' t \gg 1$ the t^3 term wins:

Key result

$$A \simeq A_0 e^{-(t/\tau_{\rm pm})^3}, \qquad \tau_{\rm pm} = \left[\frac{6}{(\nu+\eta)k_\parallel^2 v_A'^2} \right]^{1/3}$$



circles: full numerical solution, $v_A = v_{A0}(1 + 0.3\sin x)$, $\nu = \eta$

$\tau_{\rm pm} \propto (\nu+\eta)^{-1/3}$, not $(\nu+\eta)^{-1}$. Corona ($v_A = 10^3$ km/s varying over 10^3 km, $\lambda_\parallel = 10^5$ km): ~ 1 day instead of 10^7 yr (Heyvaerts & Priest 1983).

People

Historical note

- Alfvén (1942): transverse “electromagnetic-hydrodynamic” waves along the field
- Lundquist (1949): first laboratory search for Alfvén waves, in liquid metals
- Herlofson (1950): compressibility adds the fast and slow magnetosonic waves
- Friedrichs: the polar diagrams of phase and group velocity
- Belcher & Davis (1971): outward-propagating Alfvén waves in the solar wind (Mariner 5)
- Heyvaerts & Priest (1983): phase mixing as a way to heat the corona

Check your understanding

1. Why does the Alfvén wave not compress the plasma, and why does it not change $|\boldsymbol{B}|$ to first order?
2. A localised disturbance in a uniform field: where is the Alfvén energy after a time t? And the fast?
3. Why is the slow wave slower than the Alfvén wave in every direction?
4. In a low-β plasma, which wave is "sound"? How does it propagate?
5. Why is the damping rate proportional to k^2 and not $k_\parallel^2$?
6. Why does phase mixing need v_A to vary *across* the field, not along it?

Exercises in chapter 12 of the notes.

So far, and next

In this module

- linear ideal MHD $\longrightarrow$ a symmetric eigenvalue problem for $\boldsymbol{u}$
- Alfvén: $\omega = k_\parallel v_A$, transverse, tension only, $\boldsymbol{v}_g = v_A \hat{\boldsymbol{z}}$, equipartition
- fast and slow: $\omega^4 - \omega^2 k^2 (v_A^2 + c_s^2) + k^2 k_\parallel^2 v_A^2 c_s^2 = 0$; pressures in phase or antiphase; limits in θ and β
- Friedrichs diagrams; slow-mode cusps at c_T
- damping $(\nu + \eta) k^2/2$; phase mixing, $\tau_{\mathrm{pm}} \propto (\nu + \eta)^{-1/3}$

Next: module 13. When is a magnetised equilibrium stable? A rotating cylinder with a toroidal field: the Tayler and the magnetorotational instabilities.