Plasma Physics and Dynamo Theory • Module 12a

Ideal MHD: conservation laws, frozen-in field and magnetic topology

Alfio Bonanno

INAF – Osservatorio Astrofisico di Catania

Outline

1. Conservation laws: mass, momentum, energy
2. Magnetic pressure and magnetic tension
3. The frozen-in field: Alfvén's theorem, the Ω effect, the Parker spiral
4. Topology: magnetic helicity, linkage, relaxed states
5. Advection against diffusion: R_m in the laboratory and in the cosmos

Reading

Goedbloed & Poedts, ch. 4
Kulsrud, ch. 4
Moffatt (1978), ch. 2–3

Notes: chapter 12

Where we are: the ideal MHD equations (module 9b)

Key result

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{U}) = 0, \qquad \rho \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = -\nabla p + \frac{\boldsymbol{j} \times \boldsymbol{B}}{c}, \quad \boldsymbol{j} = \frac{c}{4\pi} \nabla \times \boldsymbol{B}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\rho^\gamma}\right) = 0, \qquad \frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{U} \times \boldsymbol{B}), \quad \nabla \cdot \boldsymbol{B} = 0$$

Valid for $L \gg d_i, \rho_i$ and $R_m = UL/\eta \gg 1$. $\boldsymbol{E} = -\boldsymbol{U} \times \boldsymbol{B}/c$ follows from $\boldsymbol{U}$ and $\boldsymbol{B}$.

- today: what these equations conserve, and what the conservation laws mean physically
- the answer, in one line: the field acts like elastic strings frozen into the fluid, and the strings keep their knots

Conservation laws

Momentum: from the Lorentz force to a stress tensor

Use $\boldsymbol{j} = (c/4\pi)\nabla \times \boldsymbol{B}$ and the identity $(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \nabla(B^2/2)$:

$$\frac{\boldsymbol{j} \times \boldsymbol{B}}{c} = \frac{(\boldsymbol{B} \cdot \nabla)\boldsymbol{B}}{4\pi} - \nabla \frac{B^2}{8\pi}$$

Since $\nabla \cdot \boldsymbol{B} = 0$: $(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} = \nabla \cdot (\boldsymbol{B}\boldsymbol{B})$. So the force is a divergence:

$$\frac{\boldsymbol{j} \times \boldsymbol{B}}{c} = -\nabla \cdot \mathsf{M}, \qquad \mathsf{M} = \frac{B^2}{8\pi}\mathsf{I} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi}$$

Add $\rho \, \mathrm{d}\boldsymbol{U}/\mathrm{d}t = \partial_t(\rho \boldsymbol{U}) + \nabla \cdot (\rho \boldsymbol{U}\boldsymbol{U})$ (continuity used):

Key result

$$\frac{\partial(\rho \boldsymbol{U})}{\partial t} + \nabla \cdot \left[\rho \boldsymbol{U}\boldsymbol{U} + \left(p + \frac{B^2}{8\pi}\right)\mathsf{I} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi}\right] = 0$$

Momentum is exchanged between fluid and field only through surface stresses: Maxwell's stress tensor.

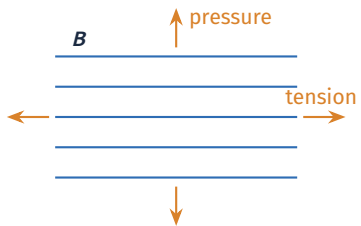
What the stress tensor means

In axes with $\hat{z} \parallel \boldsymbol{B}$:

$$\mathrm{M} = \begin{pmatrix} B^2/8\pi & 0 & 0 \\ 0 & B^2/8\pi & 0 \\ 0 & 0 & -B^2/8\pi \end{pmatrix}$$

- across $\boldsymbol{B}$: a pressure $B^2/8\pi$, pushing field lines apart
- along $\boldsymbol{B}$: a tension $B^2/8\pi$, pulling field lines straight

Field lines behave like rubber bands that repel each other. A uniform straight field exerts no force: only gradients and curvature do.



Forces on the faces of a cube of field (for the tension, the outside pulls the face outwards)

Magnetic pressure and magnetic tension

Write $\boldsymbol{B} = B\,\hat{\boldsymbol{b}}$, with ℓ the arc length along the field line:

$$(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} = B\,\frac{\partial}{\partial \ell}\left(B\hat{\boldsymbol{b}}\right) = B^2\,\frac{\partial \hat{\boldsymbol{b}}}{\partial \ell} + \hat{\boldsymbol{b}}\,\frac{\partial}{\partial \ell}\frac{B^2}{2}, \qquad \frac{\partial \hat{\boldsymbol{b}}}{\partial \ell} = \boldsymbol{\kappa} = \frac{\hat{\boldsymbol{n}}}{R_c}$$

The parallel part cancels the parallel part of $-\nabla(B^2/8\pi)$:

Key result

$$\frac{\boldsymbol{j}\times\boldsymbol{B}}{c} = -\nabla_\perp \frac{B^2}{8\pi} + \frac{B^2}{4\pi}\,\frac{\hat{\boldsymbol{n}}}{R_c}$$

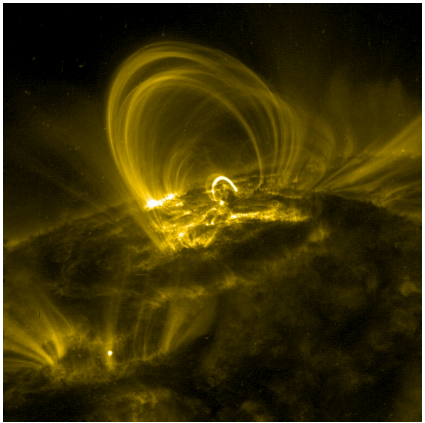
magnetic pressure gradient across the field, plus tension towards the centre of curvature, R_c the radius of curvature. No force along $\boldsymbol{B}$: the Lorentz force is always perpendicular to the field.

Plasma beta: $\beta = 8\pi p/B^2$. If $\beta \ll 1$ the field must be nearly force-free, $\boldsymbol{j}\times\boldsymbol{B} \simeq 0$: pressure and tension balance each other.

Low beta: the solar corona

Active-region corona: $n \sim 10^9$ cm^{-3}, $T \sim 10^6$ K, $B \sim 100$ G:
$\beta = 16\pi n k_B T / B^2 \approx 8\pi \times 0.28/10^4 \sim 10^{-3}$.

- the gas cannot push the field: the field is force-free, $\nabla \times \boldsymbol{B} = \alpha \boldsymbol{B}$
- the hot plasma is confined along the field lines, and traces them: coronal loops, seen in the extreme ultraviolet by SDO/AIA, Hinode, Solar Orbiter
- when the field stores too much energy it erupts: flares and coronal mass ejections (later today)



NASA/TRACE, 171 Å

Pressure balance: sunspots and the magnetopause

Sunspot (straight vertical tube, tension negligible):

$$p_{\rm in} + \frac{B^2}{8\pi} = p_{\rm out}$$

Photosphere $p_{\rm out} \approx 10^5\,\mathrm{dyn\,cm^{-2}}$: even with $p_{\rm in} = 0$, $B \le (8\pi p_{\rm out})^{1/2} \approx 1.6$ kG. Observed 2–3.5 kG: the spot is cooler and more transparent, so we see deeper, where p is larger (Wilson depression).

Magnetopause: solar wind ram pressure against the compressed dipole, $B \simeq 2B_0(R_E/r)^3$:

$$\rho_{\rm sw} V^2 = \frac{(2B_0)^2}{8\pi}\left(\frac{R_E}{r}\right)^6$$

$n = 5\,\mathrm{cm^{-3}}$, $V = 400$ km/s, $B_0 = 0.31$ G: $r \approx 10\,R_E$, as measured by every magnetospheric mission (Chapman & Ferraro 1931).

Energy (1): the magnetic energy and the Poynting flux

Faraday, $\partial_t \boldsymbol{B} = -c\nabla \times \boldsymbol{E}$, dotted with $\boldsymbol{B}/4\pi$:

$$\frac{\partial}{\partial t}\frac{B^2}{8\pi} = -\frac{c}{4\pi}\boldsymbol{B}\cdot\nabla\times\boldsymbol{E} = -\frac{c}{4\pi}\left[\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}) + \boldsymbol{E}\cdot\nabla\times\boldsymbol{B}\right] = -\nabla\cdot\boldsymbol{S} - \boldsymbol{j}\cdot\boldsymbol{E}$$

Ideal Ohm, $\boldsymbol{E} = -\boldsymbol{U}\times\boldsymbol{B}/c$:

$$\boldsymbol{j}\cdot\boldsymbol{E} = -\boldsymbol{j}\cdot\frac{\boldsymbol{U}\times\boldsymbol{B}}{c} = \boldsymbol{U}\cdot\frac{\boldsymbol{j}\times\boldsymbol{B}}{c}$$

the field loses energy at the rate at which the Lorentz force does work on the fluid. No Ohmic heating.

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B} = \frac{1}{4\pi}\left[B^2\boldsymbol{U} - (\boldsymbol{U}\cdot\boldsymbol{B})\boldsymbol{B}\right] = \frac{B^2}{4\pi}\boldsymbol{U}_\perp$$

The Poynting flux is magnetic enthalpy $B^2/4\pi$ = energy density + pressure, carried by the perpendicular flow.

Energy (2): the total energy is conserved

Kinetic energy: dot the momentum equation with $\boldsymbol{U}$. Internal energy: from $\mathrm{d}(p\rho^{-\gamma})/\mathrm{d}t = 0$.

$$\frac{\partial}{\partial t}\frac{\rho U^2}{2} + \nabla\cdot\left(\frac{\rho U^2}{2}\boldsymbol{U}\right) = -\boldsymbol{U}\cdot\nabla p + \boldsymbol{U}\cdot\frac{\boldsymbol{j}\times\boldsymbol{B}}{c}$$
$$\frac{\partial}{\partial t}\frac{p}{\gamma-1} + \nabla\cdot\left(\frac{\gamma p}{\gamma-1}\boldsymbol{U}\right) = \boldsymbol{U}\cdot\nabla p$$

Add the magnetic energy equation: the exchange terms cancel in pairs.

Key result

$$\frac{\partial}{\partial t}\left(\frac{\rho U^2}{2} + \frac{p}{\gamma-1} + \frac{B^2}{8\pi}\right) + \nabla\cdot\left[\left(\frac{\rho U^2}{2} + \frac{\gamma p}{\gamma-1}\right)\boldsymbol{U} + \frac{B^2}{4\pi}\boldsymbol{U}_\perp\right] = 0$$

With $\partial_t\boldsymbol{B} + \nabla\cdot(\boldsymbol{U}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{U}) = 0$: eight equations in conservation form, the basis of every MHD code (finite volumes, shock capturing: module 14).

The frozen-in field

Alfvén's theorem (1): the flux through a moving surface

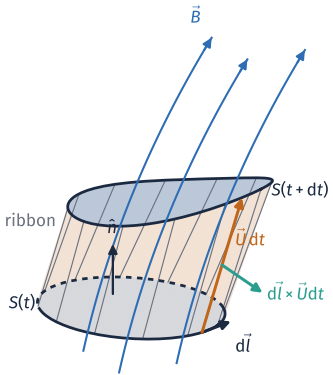
$S(t)$ moves with the fluid; its rim $C(t)$ is oriented with $\hat{n}$. $\Phi(t) = \int_{S(t)} \boldsymbol{B}(\boldsymbol{x}, t) \cdot \mathrm{d}\boldsymbol{S}$. In $\mathrm{d}t$ both $\boldsymbol{B}$ and S change:

$$\Phi(t+\mathrm{d}t) - \Phi(t) = \mathrm{d}t \int_{S(t)} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S} + \left[\int_{S(t+\mathrm{d}t)} - \int_{S(t)} \right] \boldsymbol{B}(t+\mathrm{d}t) \cdot \mathrm{d}\boldsymbol{S}$$

$S(t+\mathrm{d}t)$, $S(t)$ and the ribbon swept by C enclose the thin volume δV. Outward normals: $+\hat{n}$, $-\hat{n}$, and $\mathrm{d}\boldsymbol{l} \times \boldsymbol{U}\mathrm{d}t$ on the ribbon. Gauss:

$$\left[\int_{S(t+\mathrm{d}t)} - \int_{S(t)} \right] \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} + \mathrm{d}t \oint_C \boldsymbol{B} \cdot (\mathrm{d}\boldsymbol{l} \times \boldsymbol{U}) = \int_{\delta V} \nabla \cdot \boldsymbol{B} \, \mathrm{d}V = 0$$

The rim contributes: what the moving surface gains, the ribbon loses.



Alfvén's theorem (2): the flux is frozen in

On the ribbon, $\boldsymbol{B}(t+\mathrm{d}t) = \boldsymbol{B}(t) + O(\mathrm{d}t)$ and the area is $O(\mathrm{d}t)$: to first order use $\boldsymbol{B}(t)$. Cyclic permutation, $\boldsymbol{B} \cdot (\mathrm{d}\boldsymbol{l} \times \boldsymbol{U}) = (\boldsymbol{U} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{l}$, then Stokes on $S(t)$:

$$\left[\int_{S(t+\mathrm{d}t)} - \int_{S(t)} \right] \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = -\mathrm{d}t \oint_C (\boldsymbol{U} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{l} = -\mathrm{d}t \int_S \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{S}$$

Insert in the first line and divide by $\mathrm{d}t$:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \int_{S(t)} \left[\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) \right] \cdot \mathrm{d}\boldsymbol{S}$$

Key result

Ideal induction, $\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{U} \times \boldsymbol{B})$: $\mathrm{d}\Phi/\mathrm{d}t = 0$ for every material surface. The flux is frozen in (Alfvén 1942).

With resistivity the bracket is $\eta \nabla^2 \boldsymbol{B}$: the flux leaks on the diffusion time L^2/η.

Field lines move with the fluid

Take a thin flux tube. Its side surface carries no flux. Move it with the fluid: by Alfvén's theorem it still carries none. So it is still the side of a flux tube.

Key result

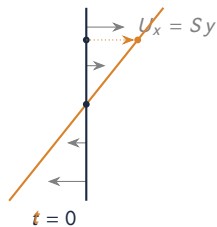
Fluid elements on the same field line stay on the same field line. Field lines can be labelled by the fluid elements, and cannot break or reconnect.

- the topology of the field (which lines are linked or knotted) is fixed while $R_m \to \infty$
- to change it you need a region where ideal MHD fails: resistivity, Hall, electron inertia (module 9b), i.e. reconnection (module 15)

Example: a star collapsing at fixed flux, $B \propto R^{-2}$. The Sun ($R_\odot$, 1 G) squeezed to a neutron star (10 km) would give $\sim 5 \times 10^9$ G; observed 10^{12} G needs stronger progenitor fields.

Example: stretching by a shear flow

Shear flow $\boldsymbol{U} = Sy\,\hat{x}$, ρ uniform: $x = x_0 + St\,y_0$, $y = y_0$, so $\partial x/\partial y_0 = St$. Initial field $\boldsymbol{B}_0 = B_0\hat{y}$: $B_x = B_0\,St$, $B_y = B_0$.



- **direction:** elements at (x_0, y_0) and $(x_0, y_0{+}\delta y_0)$ end up separated by $(St\,\delta y_0, \delta y_0)$, still on one line: $B_x/B_y = St$
- **strength:** a material segment $\delta x\,\hat{x}$ is not deformed; its flux $B_y\delta x$ (per unit z) is frozen: $B_y = B_0$
- **check:** $\boldsymbol{B}$ uniform, $\nabla\times(\boldsymbol{U}\times\boldsymbol{B}) = (\boldsymbol{B}\cdot\nabla)\boldsymbol{U} = SB_y\hat{x}$

$|\boldsymbol{B}| = B_0(1+S^2t^2)^{1/2}$: unbounded while $R_m = \infty$, fed by the flow. In a rotating body the shear is differential rotation.

Differential rotation in cylindrical coordinates

Cylindrical (s, ϕ, z), axisymmetric ($\partial_\phi = 0$): $\boldsymbol{U} = s\,\Omega(s,z)\,\hat{\boldsymbol{\phi}}$, $\boldsymbol{B} = \boldsymbol{B}_p + B_\phi \hat{\boldsymbol{\phi}}$, with poloidal part $\boldsymbol{B}_p = B_s \hat{s} + B_z \hat{z}$.

$$\boldsymbol{U} \times \boldsymbol{B} = s\Omega\,\hat{\boldsymbol{\phi}} \times (B_s \hat{s} + B_z \hat{z}) = s\Omega\,(B_z \hat{s} - B_s \hat{z}) \equiv \boldsymbol{A}, \qquad A_\phi = 0$$

Poloidal part of the induction equation: $(\nabla \times \boldsymbol{A})_s = \frac{1}{s}\partial_\phi A_z - \partial_z A_\phi = 0$, $(\nabla \times \boldsymbol{A})_z = \frac{1}{s}\partial_s(sA_\phi) - \frac{1}{s}\partial_\phi A_s = 0$. Toroidal part:

$$\begin{aligned}(\nabla \times \boldsymbol{A})_\phi = \partial_z A_s - \partial_s A_z &= \partial_z(s\Omega B_z) + \partial_s(s\Omega B_s) \\ &= s\big(B_s \partial_s \Omega + B_z \partial_z \Omega\big) + \Omega \underbrace{\big[\partial_s(sB_s) + s\,\partial_z B_z\big]}_{s\,\nabla\cdot\boldsymbol{B} = 0}\end{aligned}$$

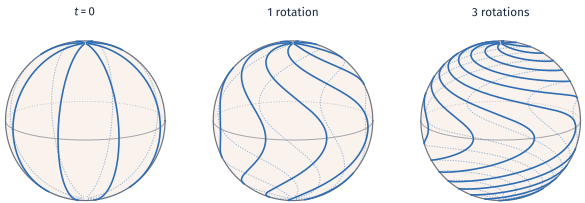
Key result

$$\frac{\partial \boldsymbol{B}_p}{\partial t} = 0, \qquad \frac{\partial B_\phi}{\partial t} = s\,(\boldsymbol{B}_p \cdot \nabla)\,\Omega$$

The Ω effect

$\boldsymbol{B}_p$ does not change, so the source of B_ϕ is constant: $B_\phi(t) = B_\phi(0) + s\,(\boldsymbol{B}_p \cdot \nabla\Omega)\,t$.

- only the gradient of Ω enters: rigid rotation does not stretch the lines
- locally it is the shear flow: in the frame rotating at $\Omega(s_0)$, $U_\phi \simeq s_0\Omega'\,(s - s_0)$, so $S = s\,\mathrm{d}\Omega/\mathrm{d}s$, $B_s \leftrightarrow B_y$, $B_\phi \leftrightarrow B_x$
- poloidal → toroidal: the Ω effect, half of the solar dynamo (modules 16–19); it stops when diffusion or the tension of B_ϕ catch up



Sun-like $\Omega = \Omega_{\mathrm{eq}}(1 - 0.25\cos^2\theta)$ (equator 25 d, poles ~35 d), seen from the equator's frame (Babcock 1961)

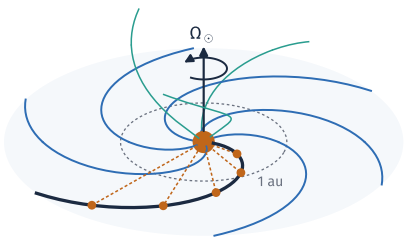
The field frozen into the solar wind: the Parker spiral

Beyond a few $R_\odot$ the wind is radial, speed V, $R_m \gg 1$. A line is anchored at a foot point that rotates with the Sun: at time t' it is at $\phi_0 + \Omega_\odot t'$ ($r = r_0$).

- a parcel leaving it at t' moves radially: $r = r_0 + V(t - t')$, $\phi = \phi_0 + \Omega_\odot t'$
- it left *on* the line, so it stays on it: the line is the locus of all parcels from one foot point

Eliminate t': $\phi - \phi_0 = \Omega_\odot \left[t - (r - r_0)/V \right]$, a spiral turning rigidly at $\Omega_\odot$. $\boldsymbol{B}$ is tangent to it:

$$\frac{B_\phi}{B_r} = r \sin\theta \, \frac{\partial \phi}{\partial r} = -\frac{\Omega_\odot r \sin\theta}{V}$$



dots: parcels, dashed: their radial paths; teal: latitude 45°

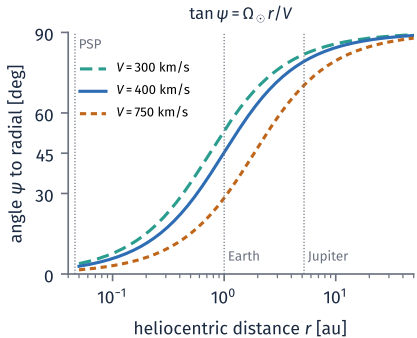
The Parker spiral: what it means

Flux through a sphere is conserved, $B_r \propto r^{-2}$, so

Key result

$$\frac{B_\phi}{B_r} = -\frac{\Omega_\odot r \sin\theta}{V}, \qquad B_r \propto \frac{1}{r^2}, \qquad B_\phi \propto \frac{1}{r}$$

- a garden sprinkler: each drop moves radially, the jet is a spiral
- $\tan\psi = \Omega_\odot r/V$: 45° at 1 au, 80° at Jupiter; far out B_ϕ dominates
- looser at high latitude ($\sin\theta$)
- Parker (1958); Mariner 2 (1962); now Wind, ACE, Parker Solar Probe, Solar Orbiter



Magnetic topology

Magnetic helicity

$\boldsymbol{B} = \nabla \times \boldsymbol{A}$, $\boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A} - \nabla\phi$. Define $H = \int_V \boldsymbol{A} \cdot \boldsymbol{B}\, \mathrm{d}V$:

$$\frac{\partial}{\partial t}(\boldsymbol{A}\cdot\boldsymbol{B}) = \partial_t \boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A}\cdot\partial_t\boldsymbol{B}, \qquad \boldsymbol{A}\cdot\partial_t\boldsymbol{B} = -c\,\boldsymbol{A}\cdot\nabla\times\boldsymbol{E} = -c\big[\boldsymbol{E}\cdot\boldsymbol{B} + \nabla\cdot(\boldsymbol{E}\times\boldsymbol{A})\big]$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -2c\int_V \boldsymbol{E}\cdot\boldsymbol{B}\,\mathrm{d}V - c\oint_{\partial V}(\phi\boldsymbol{B} + \boldsymbol{E}\times\boldsymbol{A})\cdot\mathrm{d}\boldsymbol{S}$$

Key result

Ideal MHD: $\boldsymbol{E}\cdot\boldsymbol{B} = -(\boldsymbol{U}\times\boldsymbol{B})\cdot\boldsymbol{B}/c = 0$. With $\boldsymbol{B}\cdot\hat{n} = 0$, $\boldsymbol{U}\cdot\hat{n} = 0$ on the boundary, H is conserved (Woltjer 1958).

- gauge: $\boldsymbol{A} \to \boldsymbol{A} + \nabla\chi$ changes H by $\oint \chi\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}$, zero if no field line leaves V
- cross helicity $H_c = \int \boldsymbol{U}\cdot\boldsymbol{B}\,\mathrm{d}V$ is also conserved (barotropic flow): linkage of vortex lines and field lines

Helicity counts linkage

Two thin, untwisted flux tubes with fluxes Φ_1, Φ_2.
By Stokes, $\oint_{C_1} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}$ is the flux through C_1, i.e. $\mathcal{L}\Phi_2$:

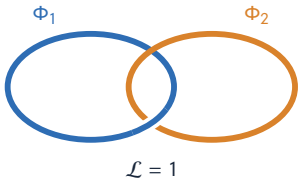
$$H = \Phi_1 \oint_{C_1} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} + \Phi_2 \oint_{C_2} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = 2\mathcal{L}\,\Phi_1\Phi_2$$

with the Gauss linking number

$$\mathcal{L} = \frac{1}{4\pi} \oint_{C_1} \oint_{C_2} \frac{(\boldsymbol{x}_1 - \boldsymbol{x}_2) \cdot (\mathrm{d}\boldsymbol{x}_1 \times \mathrm{d}\boldsymbol{x}_2)}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|^3}$$

an integer: how many times C_2 winds around C_1 (Moffatt 1969).

Twist and writhe inside a single tube count too. H is a topological invariant: flux freezing forbids unlinking.



Relaxed states: Woltjer and Taylor

Minimise the energy $W = \int B^2/8\pi \, \mathrm{d}V$ at fixed H. Vary $\boldsymbol{A}$, multiplier λ:

$$\delta W = \frac{1}{4\pi} \int \delta \boldsymbol{A} \cdot \nabla \times \boldsymbol{B} \, \mathrm{d}V, \qquad \delta H = 2 \int \delta \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V$$

$$\delta (W - \lambda H) = 0 \;\Rightarrow\; \nabla \times \boldsymbol{B} = \alpha \boldsymbol{B}, \quad \alpha = 8\pi\lambda \text{ uniform}$$

A linear force-free field (Woltjer 1958).

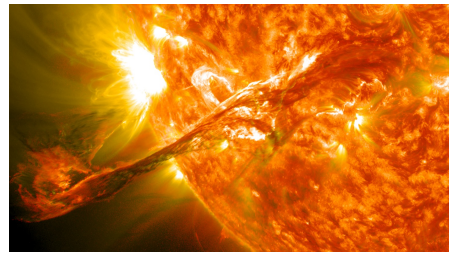
Taylor (1974): in a turbulent plasma with small η, reconnection reshuffles field lines and destroys energy faster than helicity ($\mathrm{d}H/\mathrm{d}t = -2\eta \int \boldsymbol{B} \cdot \nabla \times \boldsymbol{B}$ has one derivative less). The plasma relaxes to the Woltjer state with the same H.

Key result

In a cylinder: $B_z = B_0 J_0(\alpha r)$, $B_\theta = B_0 J_1(\alpha r)$. For $\alpha a > 2.405$, B_z reverses at the edge: the reversed-field pinch, which forms by itself.

Helicity in the laboratory and on the Sun

- **Reversed-field pinch**: ZETA (Harwell, 1960s) saw the reversal appear by itself; Taylor explained it. Today RFX-mod in Padova, the largest RFP
- **Spheromaks** relax to the Taylor state in a closed flux conserver
- **Sun**: helicity cannot be destroyed in the corona, so it accumulates. Twisted filaments and S-shaped sigmoids erupt as coronal mass ejections, which carry it into the heliosphere
- active regions: negative helicity in the north, positive in the south, in every cycle



NASA/GSFC/SDO, 31 Aug 2012

A twisted filament erupting into a CME, AIA 304 and 171 Å

Advection against diffusion

The magnetic Reynolds number in the laboratory and in the cosmos

$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}$: advection UB/L against diffusion $\eta B/L^2$,

$$R_m = \frac{UL}{\eta}, \qquad \tau_\eta = \frac{L^2}{\eta} \ \text{(diffusion time)}$$

system	η [cm^2/s]	U, L	R_m
liquid sodium (Riga, VKS)	10^3	10 m/s, 1 m	10–10^2
Earth's outer core	10^4	0.5 mm/s, 2000 km	10^3
solar convection zone	10^4	100 m/s, 10^5 km	10^{10}
interstellar medium	10^7	10 km/s, 100 pc	10^{19}

Cosmic R_m is huge: flux is frozen almost everywhere, diffusion acts only in thin sheets. Laboratory dynamos are barely above threshold.

People

Historical note

- Alfvén (1942): the frozen-in field and magnetohydrodynamic waves; Nobel prize 1970
- Parker (1958): the solar wind and its spiral field
- Babcock (1961): differential rotation winds the solar poloidal field into a toroidal one
- Woltjer (1958): magnetic helicity is an invariant; force-free fields minimise the energy
- Moffatt (1969): helicity measures the linkage of field (and vortex) lines
- Taylor (1974): relaxation at fixed helicity explains the reversed-field pinch

Check your understanding

1. Show that a uniform straight field exerts no force, and that a field with circular lines of constant B does.
2. A coronal loop has $\beta \sim 10^{-3}$. Why must its field be nearly force-free?
3. Estimate the magnetopause distance for a solar wind four times denser.
4. Why does a rigidly rotating star not wind up its poloidal field?
5. Why is the helicity of a volume gauge invariant only if no field line crosses its boundary?
6. Why does a turbulent plasma conserve its helicity better than its energy?

Exercises in chapter 12 of the notes.

So far, and next

In this module

- conservation form: Maxwell stress, Poynting flux $B^2 \boldsymbol{U}_\perp / 4\pi$
- Lorentz force = magnetic pressure + tension; β; corona, sunspots, magnetopause
- Alfvén's theorem: field lines are material lines; shear, the Ω effect, the Parker spiral
- helicity $\int \boldsymbol{A} \cdot \boldsymbol{B}$: linkage, conserved; Woltjer and Taylor states; RFP, CMEs
- R_m from 10 (laboratory) to 10^{19} (ISM)

Next: module 12b. Linear MHD waves: Alfvén, fast and slow magnetosonic waves, and how they are damped.