Plasma Physics and Dynamo Theory • Module 11

Electromagnetic waves in a cold magnetised plasma

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Outline

1. The cold-plasma dielectric tensor
2. Waves along $\boldsymbol{B}_0$: R and L waves, whistlers, Alfvén waves, Faraday rotation
3. Oblique and perpendicular waves: the general dispersion relation, O and X modes, the ionosphere
4. Measuring the Galaxy: dispersion and rotation measures

Reading

Storey (1953) on whistlers
Han et al. (2018) on pulsar rotation measures

Notes: chapter 11 (with more on the CMA diagram and on the ionosphere)

Where we are

- modules 1a, 6a: a plasma without $\boldsymbol{B}_0$ carries Langmuir waves, $\omega^2 = \omega_{pe}^2 + 3k^2 v_t^2$, and light with $\omega^2 = \omega_{pe}^2 + k^2 c^2$
- module 9b: Hall MHD gave whistlers and ion-cyclotron waves along $\boldsymbol{B}_0$

Key result

Today: all small-amplitude waves of a cold magnetised plasma, $\omega/k \gg v_t$. One dielectric tensor, three numbers R, L, P, and every mode follows.

Cold is not a small effect to neglect: it is the right limit when the waves outrun the particles, as for radio waves from pulsars, in the ionosphere and in the solar wind. Then no Landau damping (module 6a).

The cold-plasma dielectric tensor

From the particles to the waves: the plan

Maxwell's equations know about the plasma only through its current $\boldsymbol{j}$. Four steps:

1. **Motion**: each species responds to the wave field, $\boldsymbol{v}_s(\boldsymbol{E})$
2. **Current**: add up the species, $\boldsymbol{j} = \sum_s n_s q_s \boldsymbol{v}_s$, so $\boldsymbol{j}(\boldsymbol{E})$
3. **Ampère**: insert $\boldsymbol{j}(\boldsymbol{E})$; the plasma becomes a dielectric, $\boldsymbol{D}(\boldsymbol{E})$
4. **Faraday + Ampère**: eliminate $\boldsymbol{B}$; an equation for $\boldsymbol{E}$ alone, the wave equation

Uniform $\boldsymbol{B}_0 = B_0 \hat{\boldsymbol{z}}$, small perturbations $\propto e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}$, cold: no pressure. A wave exists where step 4 has a solution $\boldsymbol{E} \neq 0$.

Step 1: one species in a wave

Species s, signed gyrofrequency $\Omega_s = q_s B_0/(m_s c)$ (negative for electrons): $-i\omega\,\boldsymbol{v} = (q_s/m_s)\boldsymbol{E} + \Omega_s\,\boldsymbol{v}\times\hat{\boldsymbol{z}}$. In components, with $\boldsymbol{v}\times\hat{\boldsymbol{z}} = (v_y, -v_x, 0)$:

$$-i\omega v_x - \Omega_s v_y = \frac{q_s}{m_s}E_x, \qquad \Omega_s v_x - i\omega v_y = \frac{q_s}{m_s}E_y, \qquad -i\omega v_z = \frac{q_s}{m_s}E_z$$

The magnetic force couples v_x and v_y: a 2×2 linear system, with determinant $\Omega_s^2 - \omega^2$. Invert it:

Key result

$$v_x = \frac{iq_s}{m_s}\,\frac{\omega E_x + i\Omega_s E_y}{\omega^2 - \Omega_s^2}, \qquad v_y = \frac{iq_s}{m_s}\,\frac{\omega E_y - i\Omega_s E_x}{\omega^2 - \Omega_s^2}, \qquad v_z = \frac{iq_s}{m_s\omega}E_z$$

- a field along y moves the particle along x too: the terms in Ω_s
- $\omega \ll |\Omega_s|$: $v_x \to cE_y/B_0$, $v_y \to -cE_x/B_0$, the same for all species: the $\boldsymbol{E}\times\boldsymbol{B}$ drift of module 3a
- $\omega = |\Omega_s|$: the response diverges, a resonance; v_z does not feel $\boldsymbol{B}_0$

Step 2: the current

Insert the velocities of step 1 in $\boldsymbol{j} = \sum_s n_s q_s \boldsymbol{v}_s$; the result is linear in $\boldsymbol{E}$, $\boldsymbol{j} = \boldsymbol{\sigma} \cdot \boldsymbol{E}$:

$$\boldsymbol{\sigma} = \sum_s \frac{n_s q_s^2}{m_s} \frac{i}{\omega^2 - \Omega_s^2} \begin{pmatrix} \omega & i\Omega_s & 0 \\ -i\Omega_s & \omega & 0 \\ 0 & 0 & (\omega^2 - \Omega_s^2)/\omega \end{pmatrix} \equiv \begin{pmatrix} a & -ib & 0 \\ ib & a & 0 \\ 0 & 0 & \sigma_z \end{pmatrix}$$

- not diagonal: the off-diagonal $\mp ib$ is the Hall response of step 1, $\boldsymbol{E}$ along y drives $\boldsymbol{j}$ along x
- the form is fixed by the symmetry: $\boldsymbol{B}_0$ selects no direction in the xy plane, so $\boldsymbol{\sigma}_\perp$ must be unchanged by rotations about z: equal diagonal entries, opposite off-diagonal ones, as here

Each J_x, J_y depends on both E_x, E_y. Is there a choice of components in which each component of $\boldsymbol{j}$ is driven only by the same component of $\boldsymbol{E}$?

Circular components

Combine x and y with a factor $\pm i$: $E_\pm = E_x \pm iE_y$, $J_\pm = J_x \pm iJ_y$. From step 2:

$$J_\pm = (aE_x - ibE_y) \pm i(ibE_x + aE_y) = (a \mp b)\,E_x \pm i\,(a \mp b)\,E_y = (a \mp b)\,E_\pm.$$

With a and b from step 2, $a \mp b = \sum_s \dfrac{in_s q_s^2}{m_s} \dfrac{\omega \pm \Omega_s}{\omega^2 - \Omega_s^2}$, and the factor $\omega \pm \Omega_s$ cancels:

Key result

$$J_\pm = \sigma_\pm E_\pm, \quad J_z = \sigma_z E_z, \qquad \sigma_\pm = \sum_s \frac{i\,n_s q_s^2}{m_s(\omega \mp \Omega_s)}, \quad \sigma_z = \sum_s \frac{i\,n_s q_s^2}{m_s \omega}$$

- $E_\pm$ are the two senses of rotation of $\boldsymbol{E}$ in the xy plane (which is which: once we have the waves). A charge in $\boldsymbol{B}_0$ gyrates in a circle, so a rotating field is the natural way to push it
- each resonance now belongs to one sense of rotation: σ_+ at $\omega = \Omega_s$, σ_- at $\omega = -\Omega_s$

Step 3: Ampère with the plasma current

Ampère, $\nabla \times \boldsymbol{B} = (4\pi/c)\boldsymbol{j} + (1/c)\,\partial_t \boldsymbol{E}$, for fields $\propto e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$:

$$i\boldsymbol{k} \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} - \frac{i\omega}{c}\boldsymbol{E} = -\frac{i\omega}{c}\left(\boldsymbol{E} + \frac{4\pi i}{\omega}\boldsymbol{j}\right) \equiv -\frac{i\omega}{c}\,\boldsymbol{D}.$$

$\boldsymbol{D}$ collects the vacuum term and the plasma current. Take the circular and z components of $\boldsymbol{D} = \boldsymbol{E} + (4\pi i/\omega)\boldsymbol{j}$ and substitute $J_\pm = \sigma_\pm E_\pm$, $J_z = \sigma_z E_z$ ($\omega_{ps}^2 = 4\pi n_s q_s^2/m_s$):

$$D_\pm = \left(1 + \frac{4\pi i}{\omega}\sigma_\pm\right)E_\pm = \left[1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega \mp \Omega_s)}\right]E_\pm, \qquad D_z = \left[1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}\right]E_z$$

Key result

$$\boldsymbol{k}\times\boldsymbol{B} = -\frac{\omega}{c}\boldsymbol{D}, \qquad D_\pm = \varepsilon_\pm E_\pm, \quad D_z = P\,E_z, \qquad \varepsilon_\pm = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega \mp \Omega_s)}, \quad P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}$$

The plasma is a dielectric with one permittivity per component.

Step 4: the wave equation

Faraday, $\nabla \times \boldsymbol{E} = -(1/c)\partial_t \boldsymbol{B}$, gives $\boldsymbol{k} \times \boldsymbol{E} = (\omega/c)\boldsymbol{B}$. Apply $\boldsymbol{k}\times$ and substitute Ampère from step 3, $\boldsymbol{k} \times \boldsymbol{B} = -(\omega/c)\boldsymbol{D}$:

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}) = \frac{\omega}{c}\, \boldsymbol{k} \times \boldsymbol{B} = -\frac{\omega^2}{c^2}\boldsymbol{D}.$$

With the refractive index $\boldsymbol{n} = c\boldsymbol{k}/\omega$ and $\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{E}) = \boldsymbol{n}\,(\boldsymbol{n} \cdot \boldsymbol{E}) - n^2 \boldsymbol{E}$:

Key result

$$\boldsymbol{n}\,(\boldsymbol{n} \cdot \boldsymbol{E}) - n^2 \boldsymbol{E} + \boldsymbol{D}(\boldsymbol{E}) = 0$$

Three linear equations for $\boldsymbol{E}$; a wave exists where they have a solution $\boldsymbol{E} \neq 0$. The plasma term $\boldsymbol{D}(\boldsymbol{E})$ is diagonal in the circular and z components. The geometric term depends on the direction of $\boldsymbol{k}$:

- $\boldsymbol{k} \parallel \boldsymbol{B}_0$: also symmetric about z, also diagonal; we start here
- $\boldsymbol{k}$ oblique: $\boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{E})$ couples E_z with $E_\perp$; later in this module

Waves along the magnetic field

Along B_0: three modes

Put $\boldsymbol{n} = n\hat{\boldsymbol{z}}$ in the wave equation: $\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{E}) = n^2 E_z \hat{\boldsymbol{z}}$, so the z terms cancel and

$$\boldsymbol{D}_\perp - n^2 \boldsymbol{E}_\perp = 0, \qquad D_z = 0.$$

Take the $\pm$ components and substitute $D_\pm = \varepsilon_\pm E_\pm$, $D_z = P E_z$ from step 3:

$$(\varepsilon_- - n^2)\, E_- = 0, \qquad (\varepsilon_+ - n^2)\, E_+ = 0, \qquad P\, E_z = 0.$$

Each equation is a mode on its own:

- $n^2 = \varepsilon_-$, with only $E_- \neq 0$
- $n^2 = \varepsilon_+$, with only $E_+ \neq 0$
- $P = 0$, i.e. $\omega = \omega_p$, with only $E_z \neq 0$: $\boldsymbol{E} \parallel \boldsymbol{k} \parallel \boldsymbol{B}_0$, the Langmuir oscillation, unaffected by $\boldsymbol{B}_0$

The two transverse modes are circularly polarised. Which way does each turn, and which particles does it resonate with?

R and L waves

The particles. An electron ($q = -e$) moving along $+x$ feels $\boldsymbol{F} \propto -\hat{\boldsymbol{x}} \times \hat{\boldsymbol{z}} = +\hat{\boldsymbol{y}}$: it turns from x to y, anticlockwise seen from $+z$, right-handed about $\boldsymbol{B}_0$. The ions turn clockwise.

The field. The mode with only E_- has $E_+ = 0$, i.e. $E_y = iE_x$; at $z = 0$

$$\boldsymbol{E} = \mathrm{Re}\left[E_0(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})e^{-i\omega t}\right] = E_0(\hat{\boldsymbol{x}}\cos\omega t + \hat{\boldsymbol{y}}\sin\omega t)\,:$$

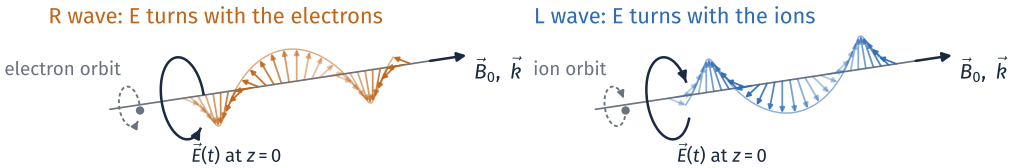
anticlockwise, with the electrons. The mode with only E_+ turns clockwise, with the ions.

The resonance. ε_- diverges at $\omega = -\Omega_s$: for electrons, $\omega = |\Omega_e|$. ε_+ diverges at $\omega = \Omega_s$: for ions, $\omega = \Omega_i$. A field turning with the particle at its gyrofrequency pushes it the same way all the time: cyclotron resonance.

Key result

Right-handed about $\boldsymbol{B}_0$: the R wave, $n_R^2 = R \equiv \varepsilon_-$, resonance at $|\Omega_e|$. Left-handed: the L wave, $n_L^2 = L \equiv \varepsilon_+$, resonance at Ω_i. Along $\boldsymbol{B}_0$: $P\,(n^2 - R)(n^2 - L) = 0$.

R and L: two circular polarisations



arrows: $\vec{E}$ along z at one instant (a helix); circle at $z = 0$: how $\vec{E}$ turns in time, anticlockwise seen from $+z$ for R

At $z = 0$, $\boldsymbol{E}_{R,L} = E_0(\hat{\boldsymbol{x}} \cos \omega t \pm \hat{\boldsymbol{y}} \sin \omega t)$; along z, at one instant, $\boldsymbol{E}$ draws a helix.

- $n_R \neq n_L$: the plasma is circularly birefringent, because the electrons follow one sense of rotation and not the other
- R and L are the eigenmodes; a linear polarisation is $\frac{1}{2}(\mathrm{R} + \mathrm{L})$, and its plane turns as they drift in phase: Faraday rotation (later in this module)

Cut-offs along B_0

For hydrogen, with $\Omega_e = -|\Omega_e|$:

$$n_R^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega - |\Omega_e|)} - \frac{\omega_{pi}^2}{\omega(\omega + \Omega_i)}, \qquad n_L^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega + |\Omega_e|)} - \frac{\omega_{pi}^2}{\omega(\omega - \Omega_i)}$$

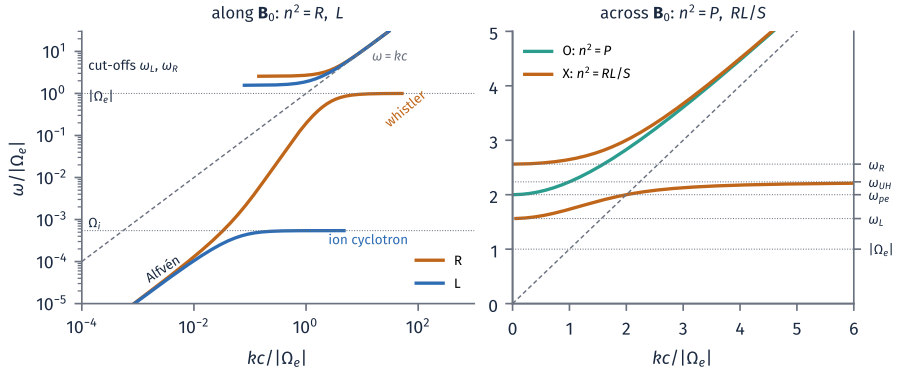
A cut-off is where $n^2 = 0$. At high frequency the ion terms are smaller by m_e/m_i, and $R = 0$ and $L = 0$ give $\omega^2 \mp \omega|\Omega_e| - \omega_{pe}^2 = 0$:

$$\omega_R = \tfrac{1}{2}\left[|\Omega_e| + (\Omega_e^2 + 4\omega_{pe}^2)^{1/2}\right], \qquad \omega_L = \tfrac{1}{2}\left[-|\Omega_e| + (\Omega_e^2 + 4\omega_{pe}^2)^{1/2}\right], \qquad \omega_L < \omega_{pe} < \omega_R.$$

Below its cut-off each wave is evanescent and is reflected.

- without $\boldsymbol{B}_0$: $\omega_R = \omega_L = \omega_{pe}$ and $n^2 = 1 - \omega_{pe}^2/\omega^2$, light in a plasma, $\omega^2 = \omega_{pe}^2 + k^2c^2$ (module 1a)
- with $\boldsymbol{B}_0$: light splits into two circular waves with different cut-offs and different phase velocities

All the cold-plasma waves in one figure



Hydrogen, $\omega_{pe} = 2|\Omega_e|$. Left, along $\boldsymbol{B}_0$: below Ω_i both waves become the Alfvén wave; between Ω_i and $|\Omega_e|$ only the R wave survives: the whistler. Right, across $\boldsymbol{B}_0$ (next section): O and X modes near the cut-offs; ions are negligible there.

Whistlers

R wave with $\Omega_i \ll \omega < |\Omega_e|$ and $\omega_{pe} \gg |\Omega_e|$: the ions do not respond, and

$$n^2 = R \simeq 1 - \frac{\omega_{pe}^2}{\omega(\omega - |\Omega_e|)} \simeq \frac{\omega_{pe}^2}{\omega|\Omega_e|} \quad \Longrightarrow \quad \omega = \frac{k^2 c^2 |\Omega_e|}{\omega_{pe}^2} = k^2 d_e^2 |\Omega_e|$$

Since $d_e^2|\Omega_e| = d_i^2\Omega_i = cB_0/(4\pi ne)$, this is the whistler of Hall MHD (module 9b), $\omega = k^2 d_i v_A$: the same wave, seen from the fluid side and from the particle side.

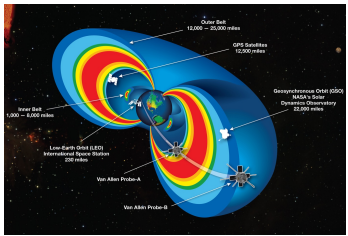
Key result

Group velocity $v_g = \mathrm{d}\omega/\mathrm{d}k = 2\omega/k = 2c\,(\omega|\Omega_e|)^{1/2}/\omega_{pe}$: high frequencies are faster: a lightning click becomes a falling tone. Along a path,

$$t(f) = \int \frac{\mathrm{d}s}{v_g} = \frac{D}{f^{1/2}}, \qquad D = \frac{1}{2c\sqrt{2\pi}} \int \frac{\omega_{pe}}{|\Omega_e|^{1/2}}\,\mathrm{d}s$$

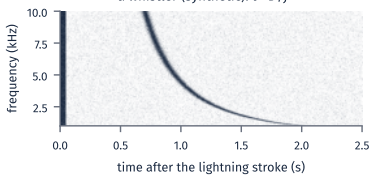
Whistlers in the magnetosphere

- Heard on field telephones in the First World War. Storey (1953): lightning launches R waves that follow the field line to the other hemisphere; D gave the first measurement of n_e in the magnetosphere
- **Chorus**: whistler waves generated by electrons injected in storms
- **Van Allen Probes** (NASA, 2012–2019): chorus accelerates belt electrons to MeV within hours (Thorne et al. 2013); hiss scatters them into the loss cone (module 4b)



NASA: the two Van Allen Probes in the radiation belts

a whistler (synthetic): $t = D/f^{1/2}$



The Alfvén limit, $\omega \ll \Omega_i$

Expand each species in ω/Ω_s: $\dfrac{1}{\omega(\omega \pm \Omega_s)} \simeq \pm\dfrac{1}{\omega\Omega_s} - \dfrac{1}{\Omega_s^2}$.

$$\sum_s \frac{\omega_{ps}^2}{\omega\Omega_s} = \frac{4\pi c}{\omega B_0}\sum_s n_s q_s = 0 \quad \text{(quasi-neutrality)}, \qquad \sum_s \frac{\omega_{ps}^2}{\Omega_s^2} = \frac{4\pi c^2}{B_0^2}\sum_s n_s m_s = \frac{c^2}{v_A^2}$$

Key result

$$R \simeq L \simeq 1 + \frac{c^2}{v_A^2} \quad \Longrightarrow \quad \omega = \frac{k v_A}{(1 + v_A^2/c^2)^{1/2}} \simeq k v_A$$

At low frequency the plasma is a dielectric with $\varepsilon \simeq c^2/v_A^2 \gg 1$: the Alfvén wave, both polarisations alike. It is the shear wave of MHD (module 12b), carried by the ion mass and the field tension. As $\omega \to \Omega_i$ it splits into the ion-cyclotron (L) and the whistler (R) branch.

Faraday rotation

Linear polarisation = R + L. Far above resonances and cut-offs ($\omega \gg \omega_{pe}, |\Omega_e|$):

$$n_{R,L} \simeq 1 - \frac{\omega_{pe}^2}{2\omega^2}\left(1 \pm \frac{|\Omega_e|}{\omega}\right) \quad \Longrightarrow \quad n_L - n_R = \frac{\omega_{pe}^2 |\Omega_e|}{\omega^3}$$

The phases drift apart; the plane of polarisation turns by half the phase difference:

$$\Delta\chi = \frac{\omega}{2c}\int (n_L - n_R)\,\mathrm{d}s = \mathrm{RM}\,\lambda^2, \qquad \mathrm{RM} = \frac{e^3}{2\pi m_e^2 c^4}\int n_e B_\parallel\,\mathrm{d}s$$

Key result

$$\mathrm{RM} = 0.812\,\frac{\mathrm{rad}}{\mathrm{m}^2}\int \frac{n_e}{\mathrm{cm}^{-3}}\,\frac{B_\parallel}{\mu\mathrm{G}}\,\frac{\mathrm{d}s}{\mathrm{pc}}$$

RM > 0 for $\boldsymbol{B}$ towards the observer; RM is the slope of χ against λ^2.

Oblique and perpendicular waves

Oblique k: back to x, y, z

For $\boldsymbol{k}$ in the xz plane at an angle θ to $\boldsymbol{B}_0$, $\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{E})$ couples E_z with E_x: no basis diagonalises both the plasma and the geometry. Write $\boldsymbol{D}$ in Cartesian components, inverting $E_\pm = E_x \pm iE_y$: $D_x = \tfrac{1}{2}(D_+ + D_-)$, $D_y = \tfrac{1}{2i}(D_+ - D_-)$, so

$$D_x = \tfrac{1}{2}\big[L(E_x + iE_y) + R(E_x - iE_y)\big] = S\,E_x - i\mathcal{D}\,E_y,$$
$$D_y = \tfrac{1}{2i}\big[L(E_x + iE_y) - R(E_x - iE_y)\big] = i\mathcal{D}\,E_x + S\,E_y,$$

with $S = \tfrac{1}{2}(R+L)$, $\mathcal{D} = \tfrac{1}{2}(R-L)$. In x, y the response is not diagonal: $\boldsymbol{v}\times\boldsymbol{B}_0$ turns a push along x into motion along y, hence the off-diagonal $\pm i\mathcal{D}$. Without $\boldsymbol{B}_0$, $R = L = P$, $\mathcal{D} = 0$: the scalar $1 - \omega_p^2/\omega^2$ of module 6a.

Key result

$$\boldsymbol{D} = \boldsymbol{\varepsilon}\cdot\boldsymbol{E}, \qquad \boldsymbol{\varepsilon} = \begin{pmatrix} S & -i\mathcal{D} & 0 \\ i\mathcal{D} & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

The wave equation for oblique k

$\boldsymbol{n} = n(\sin\theta, 0, \cos\theta)$; with $\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{E}) = \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{E}) - n^2 \boldsymbol{E}$ and $\boldsymbol{D} = \boldsymbol{\varepsilon} \cdot \boldsymbol{E}$, the wave equation reads

$$\begin{pmatrix} S - n^2\cos^2\theta & -i\mathcal{D} & n^2\sin\theta\cos\theta \\ i\mathcal{D} & S - n^2 & 0 \\ n^2\sin\theta\cos\theta & 0 & P - n^2\sin^2\theta \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0.$$

A wave exists only where the determinant vanishes: that is the dispersion relation, $n(\omega, \theta)$. The off-diagonal $n^2 \sin\theta\cos\theta$ is the coupling of E_z and E_x by the oblique $\boldsymbol{k}$; it vanishes at $\theta = 0$, where the matrix reduces to the parallel case.

The cold-plasma dispersion relation

With $N = n^2$, $s = \sin\theta$, $h = \cos\theta$, the determinant of the matrix is

$$(P - Ns^2)\left[(S - Nh^2)(S - N) - \mathcal{D}^2\right] - N^2 s^2 h^2 (S - N) = 0.$$

With $S^2 - \mathcal{D}^2 = RL$ the N^3 terms cancel, leaving a quadratic in n^2 with

$$A = S\sin^2\theta + P\cos^2\theta, \quad B = RL\sin^2\theta + PS(1 + \cos^2\theta), \quad C = PRL.$$

- two roots n^2: at each ω and θ, at most two electromagnetic modes; at $\theta = 0$, $P(n^2 - R)(n^2 - L) = 0$ as found directly
- **cut-off**, $n \to 0$ ($k \to 0$): $C = 0$, i.e. P, R or $L = 0$; the wave is reflected
- **resonance**, $n \to \infty$ ($k \to \infty$): $A = 0$; the wave is absorbed by the particles

Key result

$$An^4 - Bn^2 + C = 0$$

Across B_0: O and X modes

$\theta = \pi/2$: $A = S$, $B = RL + PS$, $C = PRL$, and the relation factorises:

$$(n^2 - P)\,(Sn^2 - RL) = 0$$

Ordinary (O) mode: $n^2 = P$
$\boldsymbol{E} \parallel \boldsymbol{B}_0$: electrons move along $\boldsymbol{B}_0$ and do not feel it. $\omega^2 = \omega_{pe}^2 + k^2c^2$, cut-off at ω_{pe}.

Extraordinary (X) mode: $n^2 = RL/S$
$\boldsymbol{E} \perp \boldsymbol{B}_0$, elliptical, partly longitudinal ($E_x/E_y = i\mathcal{D}/S$). Cut-offs ω_R, ω_L.

Key result

Resonance $S = 0$. For the electrons, the upper hybrid frequency $\omega_{UH}^2 = \omega_{pe}^2 + \Omega_e^2$: plasma oscillation plus gyration, both restoring. With ions, in a dense plasma, the lower hybrid $\omega_{LH} \simeq (|\Omega_e|\Omega_i)^{1/2}$.

Below ω_{LH} the X mode becomes the compressional (fast) MHD wave of module 12b.

The ionosphere: sounding with O and X waves

An ionosonde sends pulses upward, sweeping f from 1 to 20 MHz, and times the echoes (Breit & Tuve 1926). A wave is reflected where it meets a cut-off:

- O mode at $f = f_{pe}(h)$: the echo height maps $n_e(h)$
- X mode at $f_R \simeq f_{pe} + f_{ce}/2$, with $f_{ce} \approx 1.4$ MHz in the Earth's field

The record (ionogram) shows two traces, split by about 0.7 MHz. The highest frequency reflected by the F layer, 5–10 MHz, gives $n_e \sim 10^{5\text{–}6}$ cm^{-3}: why short-wave radio goes round the Earth.

Any angle. For $0 < \theta < \pi/2$ the modes interpolate between R/L and O/X. The CMA diagram maps, in the plane $(\omega_{pe}^2/\omega^2,\ |\Omega_e|/\omega)$, the regions bounded by the cut-offs and resonances where each mode propagates (notes).

Measuring the Galaxy

Dispersion measure

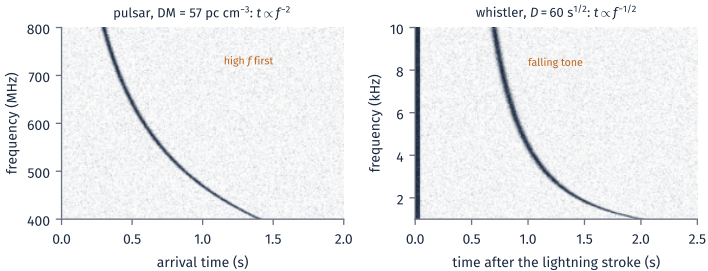
Far above ω_{pe}, magnetic corrections aside: $\omega^2 = \omega_{pe}^2 + k^2c^2$, so $v_g = c\,(1 - \omega_{pe}^2/\omega^2)^{1/2}$ and

$$t(f) = \int \frac{\mathrm{d}s}{v_g} \simeq \frac{1}{c}\int \left(1 + \frac{f_{pe}^2}{2f^2}\right)\mathrm{d}s = \frac{L}{c} + \frac{e^2}{2\pi m_e c}\frac{\mathrm{DM}}{f^2}, \qquad \mathrm{DM} = \int n_e\,\mathrm{d}s$$

Key result

$$\Delta t = 4.15\ \mathrm{ms}\ \frac{\mathrm{DM}}{\mathrm{pc\,cm^{-3}}} \times \left(\frac{f}{\mathrm{GHz}}\right)^{-2}$$

Low f arrive late. Crab pulsar: DM = 57 pc cm^{-3}.



Pulsars measure the Galactic magnetic field

The same electrons give the delay and the rotation. Divide:

Key result

$$\langle B_\parallel \rangle = \frac{\int n_e B_\parallel \, \mathrm{d}s}{\int n_e \, \mathrm{d}s} = 1.232\ \mu\mathrm{G}\ \frac{\mathrm{RM/rad\ m^{-2}}}{\mathrm{DM/pc\ cm^{-3}}}$$

- $B_\parallel$ weighted by n_e, with its sign; no distance needed
- over a thousand pulsars (Parkes, Arecibo, FAST, LOFAR, MeerKAT): a regular field of about 2 μG along the arms, reversing between some of them (Han et al. 2018)
- extragalactic sources: the Faraday sky (module 21)



CSIRO, CC BY 3.0: the Parkes (Murriyang) 64-m dish in 1969

Fast radio bursts: dispersion across the universe

- Millisecond radio flashes with DM far above the Galactic one: Lorimer et al. (2007), Parkes
- **CHIME** (Canada), 400–800 MHz, the northern sky every day: hundreds of bursts a year
- bursts with known redshift: DM grows with z, and counts the ionised "missing" baryons between galaxies (Macquart et al. 2020)
- some repeat, with RM up to 10^5 rad m^{-2}: a dense, strongly magnetised home

The same f_{pe} sets the drift of solar type III bursts (module 6b).



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People

Historical note

- Faraday (1845): the rotation of the plane of polarisation of light by a magnetic field, in glass
- Appleton (1932), Hartree (1931): the magneto-ionic theory of radio waves in the ionosphere (the Appleton–Hartree formula); Appleton, Nobel prize 1947
- Storey (1953): whistlers are lightning signals guided along the Earth's field
- Clemmow, Mullaly and Allis (1950s): one map for all cold-plasma modes, the CMA diagram
- Hewish, Bell et al. (1968): pulsars, with the first dispersion measure; Lorimer et al. (2007): fast radio bursts

Check your understanding

1. Why does the O mode not depend on $\boldsymbol{B}_0$, while the X mode does?
2. Which of R and L has a resonance at $|\Omega_e|$, and why?
3. Show that the whistler of the cold plasma and the whistler of Hall MHD are the same wave.
4. Why does a low-frequency radio pulse from a pulsar arrive later, and a low-frequency whistler too? Compare $t \propto f^{-2}$ and $t \propto f^{-1/2}$.
5. A pulsar has DM = 100 pc cm^{-3} and RM = -60 rad m^{-2}. What is $\langle B_\parallel \rangle$, and which way does it point?
6. Why is the plasma a dielectric with $\varepsilon \simeq c^2/v_A^2$ at low frequency?

Exercises in chapter 11 of the notes.

So far, and next

In this module

- cold plasma: $\boldsymbol{\varepsilon}$ from R, L, P; $An^4 - Bn^2 + C = 0$; cut-offs and resonances
- along $\boldsymbol{B}_0$: R, L; whistler $\omega = k^2 d_e^2 |\Omega_e|$; Alfvén wave below Ω_i
- across $\boldsymbol{B}_0$: O ($n^2 = P$), X ($n^2 = RL/S$), upper and lower hybrid
- DM $= \int n_e \, \mathrm{d}s$, RM $\propto \int n_e B_\parallel \, \mathrm{d}s$: $\langle B_\parallel \rangle$ of the Galaxy
- a warm plasma adds Landau damping and, across $\boldsymbol{B}_0$, the Bernstein waves at harmonics of Ω_s

Next: module 12a. Ideal MHD: conservation laws, magnetic pressure and tension, frozen-in flux, magnetic helicity.