Plasma Physics and Dynamo Theory • Module 10a

Transport: heat, momentum and magnetic flux

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Outline

1. The Chapman–Enskog expansion: closing the fluid equations with collisions
2. Heat flux and viscosity; the Spitzer conductivity $\kappa \propto T^{5/2}$
3. Magnetised transport: along and across $\boldsymbol{B}$; the corona and galaxy clusters
4. Partially ionised plasmas: ambipolar diffusion in molecular clouds, Cowling resistivity in the chromosphere

Reading

Braginskii (1965)
Spitzer (1962), ch. 5
Shu (1992), vol. II

Notes: chapter 10

Where we are

- module 9a: the moment equations need $\boldsymbol{q}$ (heat flux), $\boldsymbol{\pi}$ (viscous stress) and $\boldsymbol{R}$ (friction), which depend on the shape of f
- one way out was promised: if collisions are frequent, f is close to a local Maxwellian
- module 9b: MHD, with η from the friction; module 7a/b: $\sigma_{\parallel} = 1.96\, n e^2 \tau_e / m_e$

Key result

Today: compute $\boldsymbol{q}$ and $\boldsymbol{\pi}$ from the small deviation of f from a Maxwellian. A magnetic field makes them anisotropic; neutrals add a new kind of diffusion of the field.

These coefficients set the cooling of the corona, the viscosity and resistivity of every dynamo ($P_m = \nu/\eta$), and the time a molecular cloud needs to form a star.

The Chapman–Enskog expansion

The small parameter: the Knudsen number

Mean free path of the electrons, $\lambda_{\mathrm{mfp}} = v_{te}\tau_e$, with $\tau_e = 1/\nu_{ei}$ (module 2a):

$$\lambda_{\mathrm{mfp}} \simeq 1.4 \times 10^{13} \frac{T_{\mathrm{eV}}^2}{n \ln \Lambda} \text{ cm} \quad \propto \frac{T^2}{n}$$

Knudsen number $\mathrm{Kn} = \lambda_{\mathrm{mfp}}/L$. Expand

$$f = f_M(n, \boldsymbol{u}, T) + f_1 + \ldots, \qquad f_1 \sim \mathrm{Kn}\, f_M$$

n, $\boldsymbol{u}$, T are carried entirely by f_M: f_1 has no density, momentum or energy of its own.

	λ_{mfp}	Kn
corona, 100 eV	80 km	10^{-2}
solar wind, 1 au	0.7 au	~ 1
cluster core	0.3 kpc	0.03
cluster outskirts	10 kpc	~ 0.1
tokamak core	10 km	10^4

n [cm^{-3}], T, L: corona 10^9, 100 eV, 10^4 km; wind 5, 10 eV, 1 au; core 10^{-2}, 5 keV, 10 kpc; outskirts 10^{-3}, 8 keV, 100 kpc; tokamak 10^{14}, 10 keV, 1 m.

The corona is collisional along $\boldsymbol{B}$, but only by a factor of about 100.

Order by order

Model collision operator: relaxation to f_M in a time $\tau = 1/\nu$,

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \boldsymbol{a} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = -\nu \left(f - f_M \right)$$

- order Kn^{-1}: $C(f_0) = 0$, so $f_0 = f_M$: the ideal fluid, $\boldsymbol{q} = 0$, $\boldsymbol{\pi} = 0$ (Euler)
- order Kn^0: the left side acting on f_M is balanced by collisions,

$$f_1 = -\tau \left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla + \boldsymbol{a} \cdot \frac{\partial}{\partial \boldsymbol{v}} \right) f_M,$$

  with $\partial n/\partial t$, $\partial \boldsymbol{u}/\partial t$, $\partial T/\partial t$ taken from the Euler equations

Key result

The gradients of the ideal flow drive f_1; f_1 gives $\boldsymbol{q}$ and $\boldsymbol{\pi}$: Fourier and Navier–Stokes, first order in Kn.

Computing the drive, step by step (1)

With $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$: $\ln f_M = \ln n - \frac{3}{2}\ln T - \dfrac{mw^2}{2T} + \text{const}$, and $\partial_t + \boldsymbol{v}\cdot\nabla = \dfrac{\mathrm{d}}{\mathrm{d}t} + \boldsymbol{w}\cdot\nabla$.

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \boldsymbol{w}\cdot\nabla\right)\ln n = -\nabla\cdot\boldsymbol{u} + \boldsymbol{w}\cdot\nabla\ln n$$

$$-\frac{3}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t} + \boldsymbol{w}\cdot\nabla\right)\ln T = +\nabla\cdot\boldsymbol{u} - \frac{3}{2}\,\boldsymbol{w}\cdot\nabla\ln T \qquad \left(\text{Euler: } \frac{\mathrm{d}\ln T}{\mathrm{d}t} = -\frac{2}{3}\nabla\cdot\boldsymbol{u}\right)$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \boldsymbol{w}\cdot\nabla\right)\left(-\frac{mw^2}{2T}\right) = \frac{mw^2}{2T}\left(-\frac{2}{3}\nabla\cdot\boldsymbol{u} + \boldsymbol{w}\cdot\nabla\ln T\right) + \frac{m}{T}\,\boldsymbol{w}\cdot\left(\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \boldsymbol{w}\cdot\nabla\boldsymbol{u}\right)$$

$$\boldsymbol{a}\cdot\frac{\partial}{\partial\boldsymbol{v}}\ln f_M = -\frac{m}{T}\,\boldsymbol{w}\cdot\boldsymbol{a}$$

Euler momentum: $\mathrm{d}\boldsymbol{u}/\mathrm{d}t = -\nabla p/(mn) + \boldsymbol{a}$. The force $\boldsymbol{a}$ cancels, and $\frac{m}{T}\boldsymbol{w}\cdot\nabla p/(mn) = \boldsymbol{w}\cdot(\nabla\ln n + \nabla\ln T)$.

Computing the drive, step by step (2)

Add the four lines. $\nabla \ln n$ cancels; $\nabla \ln T$ appears with $\frac{mw^2}{2T} - \frac{3}{2} - 1$; the $\nabla \cdot \boldsymbol{u}$ terms cancel except $-\frac{mw^2}{3T}\nabla \cdot \boldsymbol{u}$, which makes the tensor traceless:

Key result

$$f_1 = -\tau f_M \left[\left(\frac{mw^2}{2T} - \frac{5}{2} \right) \boldsymbol{w} \cdot \nabla \ln T + \frac{m}{T} \left(w_i w_j - \frac{w^2}{3} \delta_{ij} \right) \frac{\partial u_i}{\partial x_j} \right]$$

odd in $\boldsymbol{w}$, driven by ∇T: fast particles from the hot side, slow from the cold. The $-\frac{5}{2}$ makes $\int \boldsymbol{w} f_1 \, \mathrm{d}^3 v = 0$: energy flows, particles do not.

even, traceless, driven by the shear of $\boldsymbol{u}$: f stretched along the strain, an anisotropic pressure.

No ∇n, no force, no $\nabla \cdot \boldsymbol{u}$: an isothermal gas in hydrostatic balance, or one compressed uniformly, stays Maxwellian.

The heat flux

$\boldsymbol{q} = \int \frac{1}{2} m w^2 \, \boldsymbol{w} \, f_1 \, \mathrm{d}^3 v$. Isotropy gives $\langle w_i w_j g(w^2) \rangle = \frac{1}{3}\delta_{ij} \langle w^2 g \rangle$, and for a Maxwellian $\langle w^4 \rangle = 15 v_t^4$, $\langle w^6 \rangle = 105 v_t^6$:

$$\boldsymbol{q} = -\frac{\tau n}{3}\left[\frac{m^2}{4T}\,105 v_t^6 - \frac{5m}{4}\,15 v_t^4\right]\frac{\nabla T}{T} = -\frac{\tau n m v_t^4}{3}\cdot\frac{30}{4}\cdot\frac{\nabla T}{T} = -\frac{5}{2}\,\frac{nT\tau}{m}\,\nabla T$$

Key result: Fourier's law from kinetic theory

$$\boldsymbol{q} = -\kappa \nabla T, \qquad \kappa = \frac{5}{2}\,\frac{nT\tau}{m} = \frac{5}{2}\, n v_t \lambda_{\mathrm{mfp}} \qquad \text{(exact: 3.16 for electrons)}$$

With $\tau_e \propto T^{3/2}/n$, the density cancels:

$$\kappa_{\parallel} \simeq 1.8 \times 10^{-5}\,\frac{T^{5/2}}{\ln \Lambda}\ \mathrm{erg\,s^{-1}\,cm^{-1}\,K^{-1}} \qquad \text{(Spitzer–Härm)}$$

Electrons carry the heat: $\kappa_i/\kappa_e \sim (m_e/m_i)^{1/2}$.

The viscous stress

$\pi_{ij} = \int m\, w_i w_j f_1 \, \mathrm{d}^3 v$, with $\langle w_i w_j w_k w_l \rangle = v_t^4 (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$:

Key result

$$\pi_{ij} = -\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij} \nabla \cdot \boldsymbol{u} \right), \qquad \mu = p\,\tau$$

- ions dominate: $\mu_i = 0.96\, nT\tau_i$, and $\tau_i \simeq (2m_i/m_e)^{1/2}\tau_e$
- kinematic viscosity $\nu = \mu/\rho \sim v_{ti}\lambda_i$, again $\propto T^{5/2}/n$
- for a dynamo, the ratio that matters is the magnetic Prandtl number $P_m = \nu/\eta \propto T^4/n$ (module 18):

	base of the solar convection zone	corona	warm ISM	cluster
P_m	$\sim 10^{-1}$	$\sim 10^{10}$	$\sim 10^{11}$	$\sim 10^{27}$

When the expansion fails

- Fourier's law assumes Kn $\ll 1$. It gives $q \simeq \frac{5}{2}\, n T v_t\, (\lambda_{\rm mfp}/L_T)$, which exceeds the largest possible flux, all particles streaming, $q_{\rm sat} \sim n T v_t$, when $\lambda_{\rm mfp} \gtrsim L_T/2$
- the heat is carried by electrons of 2–3 v_{te}, whose mean free path is $(2\text{–}3)^4 \simeq 10\text{–}80$ times longer: the expansion fails earlier than Kn suggests
- then: a saturated flux $q \simeq 0.4\, n T\, (2T/\pi m_e)^{1/2}$ (a limiter), or Landau-fluid closures that mimic the kinetic response (module 9a), or kinetic theory

Where it matters

Solar flares and the corona–chromosphere transition region (steep ∇T); evaporation of cold clouds in hot gas; the solar wind, where the electron heat flux is limited by kinetic instabilities rather than collisions.

Magnetised transport

Across the field the step is the gyroradius

Random walk: along $\boldsymbol{B}$ the step is λ_{mfp}, across $\boldsymbol{B}$ it is the gyroradius $\rho = v_t/\Omega$:

$$D_\parallel \sim \frac{\lambda_{\mathrm{mfp}}^2}{\tau} = v_t^2 \tau, \qquad D_\perp \sim \frac{\rho^2}{\tau} = \frac{v_t^2}{\Omega^2 \tau}, \qquad \frac{D_\perp}{D_\parallel} \sim \frac{1}{(\Omega\tau)^2}$$

Fluid version, isothermal and steady, friction $-mn\nu\boldsymbol{u}$: $0 = -T\nabla n + (qn/c)\,\boldsymbol{u}\times\boldsymbol{B} - mn\nu\,\boldsymbol{u}$. Along $\boldsymbol{B}$: $n\boldsymbol{u}_\parallel = -D\nabla_\parallel n$, $D = T/m\nu$. Across, take $\times\hat{\boldsymbol{b}}$ and solve the 2×2 system:

Key result

$$n\boldsymbol{u}_\perp = -\frac{D}{1+\Omega^2\tau^2}\,\nabla_\perp n \;+\; \frac{\Omega\tau D}{1+\Omega^2\tau^2}\,\hat{\boldsymbol{b}}\times\nabla n$$

$\Omega = qB/mc$ (signed). For $\Omega\tau \gg 1$ diffusion drops by $(\Omega\tau)^{-2}$; the second term tends to $(cT/qB)\,\hat{\boldsymbol{b}}\times\nabla n$, the collisionless diamagnetic flux of module 9a.

Braginskii's coefficients

Heat flux in a magnetised plasma, $\hat{\boldsymbol{b}} = \boldsymbol{B}/B$: $\quad \boldsymbol{q} = -\kappa_\parallel \hat{\boldsymbol{b}}\,(\hat{\boldsymbol{b}} \cdot \nabla T) - \kappa_\perp \nabla_\perp T - \kappa_\wedge \hat{\boldsymbol{b}} \times \nabla T$

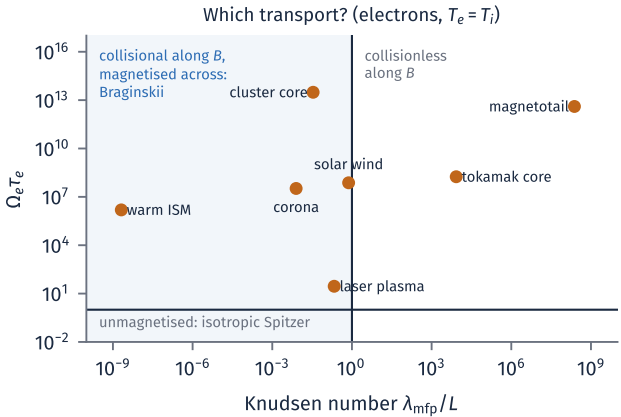
	electrons	ions	
$\kappa_\parallel$	$3.16\, nT\tau_e/m_e$	$3.9\, nT\tau_i/m_i$	(smaller by $(m_e/m_i)^{1/2}$)
$\kappa_\perp$	$4.66\, nT/(m_e\Omega_e^2\tau_e)$	$2\, nT/(m_i\Omega_i^2\tau_i)$	(ions dominate)
$\kappa_\wedge$	$\frac{5}{2}\, nT/(m_e\Omega_e)$	$\frac{5}{2}\, nT/(m_i\Omega_i)$	(no τ: diamagnetic)

Valid if Kn $\ll 1$ along $\boldsymbol{B}$. Viscosity: same structure, five coefficients; friction: $\eta_\parallel = 0.51\,\eta_\perp$ (module 9a).

Key result

$$\frac{\kappa_\perp}{\kappa_\parallel} \sim \frac{1}{(\Omega_e\tau_e)^2}\left(\frac{m_i}{m_e}\right)^{1/2} : \quad 10^{-12} \text{ in the corona, } 10^{-24} \text{ in a cluster}$$

Which transport where?

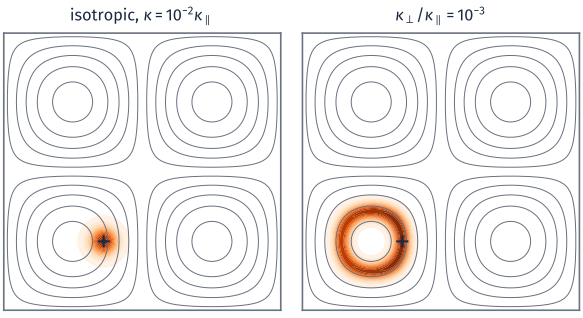


Plasmas of module 9b, with λ_{mfp} and $\Omega_e \tau_e$ from their n, T, B.

Nearly all are magnetised ($\Omega_e \tau_e \gg 1$).

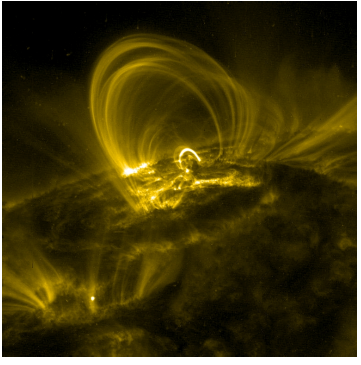
Only the ISM, the corona and cluster cores are collisional along $\boldsymbol{B}$; the solar wind, the magnetosphere and fusion plasmas are not: their transport is kinetic (and, in tokamaks, neoclassical or turbulent).

Heat follows the field lines



gray: field lines; colour: temperature

Same hot spot, same time. With $\kappa_{\perp} \ll \kappa_{\parallel}$, T spreads along the field line through the spot.



NASA/TRACE, 171 Å

Coronal loops: bundles of field lines, each at its own T, sharp across $\boldsymbol{B}$.

The corona: conduction against radiation

A loop of half-length $L = 10^4$ km, $n = 10^9$ cm^{-3}, $T = 100$ eV (1.2×10^6 K):

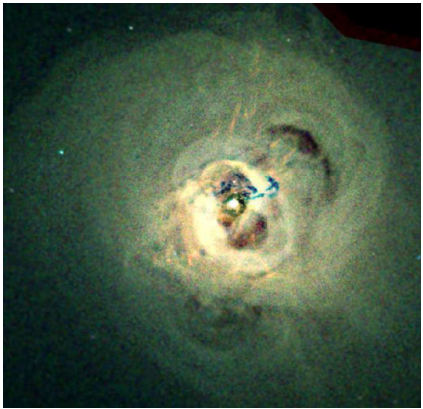
$$t_{\rm cond} \simeq \frac{3 n k_B T\, L^2}{\kappa_\parallel T} \approx 300\ {\rm s}, \qquad t_{\rm rad} \simeq \frac{3 n k_B T}{n^2 \Lambda(T)} \approx 5000\ {\rm s} \quad (\Lambda \approx 10^{-22}\ {\rm erg\,cm^3\,s^{-1}})$$

- conduction is fast along the loop and nothing across it: the heat deposited in the corona is carried down the legs to the dense transition region, where it is radiated
- balancing heating, conduction and radiation gives the loop scaling law $T_{\rm max} \simeq 1.4 \times 10^3\ (pL)^{1/3}$ K (cgs), about 10^6 K for these numbers (Rosner, Tucker & Vaiana 1978)

Hinode, SDO, Solar Orbiter

X-ray and EUV images measure T and n along loops: the profiles test the $T^{5/2}$ law and locate the heating.

Galaxy clusters: conduction switched off across the field



NASA/CXC/IoA/A. Fabian et al.

Hot gas, $T \sim 5$ keV, $n \sim 10^{-3}$–10^{-2} cm^{-3}: $\lambda_{\rm mfp} \sim 0.3$–$10$ kpc.

Cold fronts (Chandra): sharp edges between gas at different T, narrower than a few kpc, so narrower than $\lambda_{\rm mfp}$. Spitzer conduction would erase them in a few Myr, much less than their age.

Explanation: the field is draped along the front, and $\kappa_\perp/\kappa_\parallel \sim 10^{-24}$ (Vikhlinin, Markevitch & Murray 2001).

Along $\boldsymbol{B}$ too, conduction is below Spitzer: at $\beta \sim 100$ kinetic instabilities scatter the electrons.

Partially ionised plasmas

Three fluids: electrons, ions, neutrals

Neutrals feel no Lorentz force, only collisions with the ions. Ions plus electrons (no inertia or pressure; electrons too light for friction with neutrals), $\boldsymbol{E}$ cancels: $0 = \boldsymbol{j} \times \boldsymbol{B}/c - \rho_i \nu_{in} (\boldsymbol{u}_i - \boldsymbol{u}_n)$. The magnetic force pushes the ions (and the electrons tied to them) through the neutrals:

Key result

$$\boldsymbol{v}_D \equiv \boldsymbol{u}_i - \boldsymbol{u}_n = \frac{\boldsymbol{j} \times \boldsymbol{B}}{c\,\rho_i \nu_{in}} = \frac{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi \rho_i \nu_{in}}$$

	$x = n_i/n_n$	
molecular cloud core	$\sim 10^{-7}$	cosmic rays ionise; $x \propto n^{-1/2}$
solar chromosphere	10^{-4}–10^{-1}	$T \simeq 6000$–10^4 K
protoplanetary disk (dead zone)	$< 10^{-12}$	

Ambipolar diffusion

The field is frozen into the ions and electrons: $\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{u}_i \times \boldsymbol{B})$. With $\boldsymbol{u}_i = \boldsymbol{u}_n + \boldsymbol{v}_D$:

Key result

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u}_n \times \boldsymbol{B}) + \nabla \times \left[\frac{\big((\nabla \times \boldsymbol{B}) \times \boldsymbol{B}\big) \times \boldsymbol{B}}{4\pi \rho_i \nu_{in}} \right]$$

- ambipolar diffusivity $\eta_{\mathrm{AD}} = \dfrac{B^2}{4\pi \rho_i \nu_{in}} = \dfrac{v_{A,i}^2}{\nu_{in}}$: nonlinear in B
- it acts only on the part of $\boldsymbol{j}$ perpendicular to $\boldsymbol{B}$: a force-free field ($\boldsymbol{j} \parallel \boldsymbol{B}$) does not diffuse
- "ambipolar": ions and electrons move together; the field slips through the neutrals

Time to cross a scale L: $t_{\mathrm{AD}} = L/v_D \simeq 4\pi \rho_i \nu_{in} L^2/B^2 = L^2/\eta_{\mathrm{AD}}$.

How long does a cloud core take to lose its field?

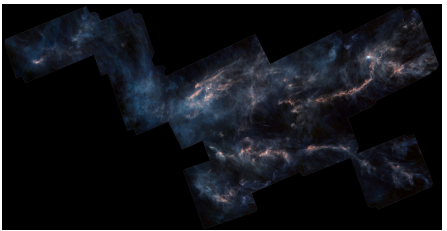
Core: $n_{\mathrm{H_2}} = 10^4$ cm^{-3} ($\rho = 3.9 \times 10^{-20}$ g cm^{-3}), $B = 30\,\mu$G, $L = 0.1$ pc.
Cosmic-ray ionisation: $\rho_i \simeq C\rho^{1/2}$, $C \simeq 3 \times 10^{-16}$ (cgs): $\rho_i \approx 6 \times 10^{-26}$ g cm^{-3}, $x \approx 10^{-7}$.
Drag: $\rho_i v_{in} = \gamma \rho_i \rho$, $\gamma \simeq 3.5 \times 10^{13}$ cm^3 g^{-1} s^{-1}.

$$t_{\mathrm{AD}} = \frac{4\pi\gamma\rho_i\rho L^2}{B^2} \approx 3\ \mathrm{Myr}$$

against $t_{\mathrm{ff}} = (3\pi/32G\rho)^{1/2} \approx 0.3$ Myr.

Key result

$t_{\mathrm{AD}} \approx 10\,t_{\mathrm{ff}}$: a magnetised core leaks its support slowly, then collapses.



ESA/Herschel/PACS, SPIRE (CC BY-SA 3.0 IGO)

Taurus. Dust polarisation (Planck, ALMA) shows $\boldsymbol{B}$ perpendicular to the dense filaments, parallel to the striations feeding them; Zeeman splitting measures B in cores.

The chromosphere: Cowling resistivity

One-fluid Ohm's law for a partially ionised gas: an extra resistance for the current across $\boldsymbol{B}$,

$$\boldsymbol{E} + \frac{\boldsymbol{u} \times \boldsymbol{B}}{c} = \frac{\boldsymbol{j}}{\sigma} + \frac{\xi_n^2 B^2}{c^2 \rho_i \nu_{in}} \boldsymbol{j}_\perp + \ldots, \qquad \xi_n = \frac{\rho_n}{\rho}$$

- $\boldsymbol{j}_\parallel$ sees σ; $\boldsymbol{j}_\perp$ sees the Cowling resistivity $\eta_C = \eta + \xi_n^2 \eta_{\mathrm{AD}}$; its heating is ion–neutral friction

At about 1000 km ($n_n \sim 10^{14}$ cm^{-3}, $x \sim 10^{-3}$, $T = 6000$ K, $B = 100$ G):

$$\eta \approx 3 \times 10^7 \text{ cm}^2\text{s}^{-1}, \qquad \eta_{\mathrm{AD}} \approx 6 \times 10^9 \text{ cm}^2\text{s}^{-1} \quad (\sim 200\times)$$

Hinode, IRIS, DKIST

Ion–neutral friction damps Alfvén waves, heats the gas and lets the field slip in spicules and small-scale reconnection; ion and neutral lines are compared to detect the drift.

People

Historical note

- Chapman (1916) and Enskog (1917): transport coefficients of a gas from the Boltzmann equation, independently
- Spitzer & Härm (1953): conductivity and heat flux of a fully ionised gas
- Mestel & Spitzer (1956): ambipolar diffusion lets magnetic flux leave a collapsing cloud
- Cowling (1957): the resistivity of a partially ionised gas
- Braginskii (1958, review 1965): transport in a magnetised two-fluid plasma
- Rosner, Tucker & Vaiana (1978): the scaling law of coronal loops

Check your understanding

1. Why does f_1 contain no term in ∇n or in the electric field?
2. Why is the heat conductivity of a plasma independent of the density? Is the electrical conductivity?
3. Why do electrons dominate heat conduction along $\boldsymbol{B}$ but ions across it?
4. Estimate $\Omega_e \tau_e$ in a coronal loop with $B = 10$ G. What does it imply for the shape of loops?
5. Why does ambipolar diffusion not act on a force-free field?
6. How does t_{AD} scale with the density of a cloud if $x \propto n^{-1/2}$ and B is fixed?

Exercises in chapter 10 of the notes.

So far, and next

In this module

- Chapman–Enskog: $f = f_M + f_1$, f_1 driven by ∇T and the shear
- $\boldsymbol{q} = -\kappa \nabla T$, $\kappa \propto T^{5/2}$, independent of n; $\mu = p\tau$; $P_m \propto T^4/n$
- magnetised: $\kappa_\perp/\kappa_\parallel \sim (\Omega\tau)^{-2}$; heat follows field lines (loops, cold fronts)
- neutrals: ambipolar diffusion, $\eta_{\mathrm{AD}} = B^2/4\pi\rho_i\nu_{in}$; $t_{\mathrm{AD}} \approx 10\, t_{\mathrm{ff}}$ in cores; Cowling resistivity

Next: module 11. Electromagnetic waves in a cold magnetised plasma: whistlers, Faraday rotation, and how pulsars measure the Galactic field.