Plasma Physics and Dynamo Theory • Module 9b

The generalised Ohm's law and the MHD limit

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Outline

1. One fluid from two
2. The generalised Ohm's law, term by term
3. The MHD limit, and why it works without collisions
4. Hall MHD: reconnection in space and in the laboratory
5. The Biermann battery: magnetic fields from nothing

Reading

Goedbloed & Poedts, ch. 3–4
Kulsrud, ch. 3, 13
Yamada, Kulsrud & Ji (2010)

Notes: chapter 9

Where we are

- module 9a: two fluids, electrons and ions, each with continuity, momentum, energy, coupled by friction $\boldsymbol{R}_{ei} = (ne/\sigma)\boldsymbol{j}$ and by $\boldsymbol{E}$, $\boldsymbol{B}$
- today: add them to get one fluid, and subtract them to get Ohm's law

Key result

Ohm's law decides how the magnetic field moves: with the fluid (ideal MHD), slipping through it (resistive), with the electrons only (Hall), or created from nothing (battery).

This is the equation that the rest of the course, and all of dynamo theory, rests on.

One fluid from two

One-fluid variables

Electrons ($-e$) and ions (Ze, mass m_i):

$$\rho = m_i n_i + m_e n_e, \qquad \boldsymbol{U} = \frac{m_i n_i \boldsymbol{u}_i + m_e n_e \boldsymbol{u}_e}{\rho}, \qquad \boldsymbol{j} = Z e n_i \boldsymbol{u}_i - e n_e \boldsymbol{u}_e$$

$\boldsymbol{U}$: centre-of-mass velocity, essentially the ion velocity. $\boldsymbol{j}$: the relative motion.

Quasi-neutral, $n_e = Z n_i \equiv n$; to first order in m_e/m_i, invert:

Key result

$$\boldsymbol{u}_i \simeq \boldsymbol{U}, \qquad \boldsymbol{u}_e \simeq \boldsymbol{U} - \frac{\boldsymbol{j}}{ne}$$

In the centre-of-mass frame the electrons carry the current. Everything that follows comes from substituting this $\boldsymbol{u}_e$ into the electron equation.

Adding the two momentum equations

Sum over s of $m_s n_s \, \mathrm{d}_s \boldsymbol{u}_s / \mathrm{d}t = -\nabla \cdot \mathrm{P}_s + q_s n_s (\boldsymbol{E} + \boldsymbol{u}_s \times \boldsymbol{B}/c) + \boldsymbol{R}_s$:

- friction: $\boldsymbol{R}_{ei} + \boldsymbol{R}_{ie} = 0$ (action and reaction)
- electric force: $\sum_s q_s n_s \boldsymbol{E} = 0$ (quasi-neutral)
- magnetic force: $\sum_s q_s n_s \boldsymbol{u}_s \times \boldsymbol{B}/c = \boldsymbol{j} \times \boldsymbol{B}/c$
- inertia: $\sum m_s n_s \mathrm{d}_s \boldsymbol{u}_s / \mathrm{d}t \simeq \rho \, \mathrm{d}\boldsymbol{U}/\mathrm{d}t$ (electron inertia and the relative-motion stress are smaller by m_e/m_i)

Key result

$$\rho \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = -\nabla p + \frac{\boldsymbol{j} \times \boldsymbol{B}}{c}, \qquad p = p_e + p_i \qquad \text{(isotropic, no viscosity)}$$

And the continuity equations, multiplied by m_s and added: $\partial_t \rho + \nabla \cdot (\rho \boldsymbol{U}) = 0$.

The generalised Ohm’s law

The electron momentum equation, solved for $\boldsymbol{E}$

Electrons, with $\boldsymbol{R}_{ei} = (ne/\sigma)\boldsymbol{j}$ (9a):

$$m_e n \frac{\mathrm{d}_e \boldsymbol{u}_e}{\mathrm{d}t} = -\nabla p_e - en\left(\boldsymbol{E} + \frac{\boldsymbol{u}_e \times \boldsymbol{B}}{c}\right) + \frac{ne}{\sigma}\boldsymbol{j}$$

Divide by $-en$ and move $\boldsymbol{E}$ to the left:

$$\boldsymbol{E} + \frac{\boldsymbol{u}_e \times \boldsymbol{B}}{c} = \frac{\boldsymbol{j}}{\sigma} - \frac{\nabla p_e}{en} - \frac{m_e}{e}\frac{\mathrm{d}_e \boldsymbol{u}_e}{\mathrm{d}t}$$

Insert $\boldsymbol{u}_e = \boldsymbol{U} - \boldsymbol{j}/(ne)$: the magnetic term splits,

$$\frac{\boldsymbol{u}_e \times \boldsymbol{B}}{c} = \frac{\boldsymbol{U} \times \boldsymbol{B}}{c} - \frac{\boldsymbol{j} \times \boldsymbol{B}}{nec}$$

and in the inertia term keep the part linear in $\boldsymbol{j}$: $-(m_e/e)\,\mathrm{d}_e \boldsymbol{u}_e/\mathrm{d}t \simeq (m_e/ne^2)\,\partial_t \boldsymbol{j}$.

The generalised Ohm's law

Key result

$$\boldsymbol{E} + \frac{\boldsymbol{U} \times \boldsymbol{B}}{c} = \underbrace{\frac{\boldsymbol{j}}{\sigma}}_{\text{resistive}} + \underbrace{\frac{\boldsymbol{j} \times \boldsymbol{B}}{nec}}_{\text{Hall}} - \underbrace{\frac{\nabla p_e}{ne}}_{\text{electron pressure}} + \underbrace{\frac{m_e}{ne^2}\frac{\partial \boldsymbol{j}}{\partial t}}_{\text{electron inertia}}$$

- left: the electric field in the frame moving with the fluid
- right: everything that lets the plasma differ from a perfect conductor moving with $\boldsymbol{U}$
- ideal MHD: the right-hand side is zero, $\boldsymbol{E} = -\boldsymbol{U} \times \boldsymbol{B}/c$

Not included: thermal force ($-0.71\nabla_{\parallel} T_e$), anisotropic P_e, the ion terms of order m_e/m_i. Each can matter somewhere; the four above are the ones that change how $\boldsymbol{B}$ moves.

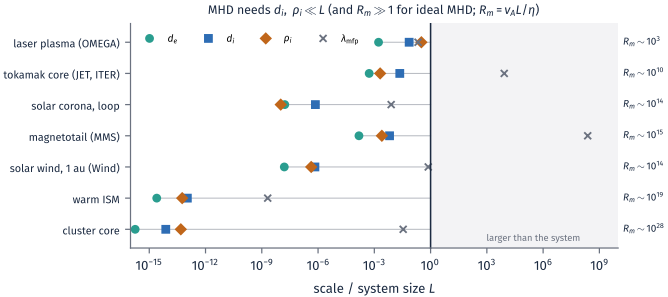
How big is each term?

Compare each with UB/c, for flows U on a scale L, $j \sim cB/4\pi L$ (Ampère), times L/U:

term	ratio to UB/c	important when
resistive	$\dfrac{\eta}{UL} = \dfrac{1}{R_m}$, $\eta = \dfrac{c^2}{4\pi\sigma}$	$R_m \lesssim 1$
Hall	$\dfrac{v_A}{U}\dfrac{d_i}{L}$, $d_i = \dfrac{c}{\omega_{pi}}$	$L \lesssim d_i$
electron pressure	$\dfrac{c_s}{U}\dfrac{\rho_s}{L}$, $\rho_s = \dfrac{c_s}{\Omega_i}$	$L \lesssim \rho_s$
electron inertia	$\dfrac{d_e^2}{L^2}$, $d_e = \dfrac{c}{\omega_{pe}}$	$L \lesssim d_e$

Hall, for example: $\dfrac{jB/nec}{UB/c} = \dfrac{j}{neU} = \dfrac{cB}{4\pi neLU}$, and $\dfrac{cB}{4\pi ne} = v_A d_i$ (check: $v_A = B/\sqrt{4\pi n m_i}$, $d_i = c\sqrt{m_i/4\pi ne^2}$).

Scales across the plasma universe



Representative n, T, B, L; Spitzer η; R_m with $U = v_A$. From a laser target (10^{-2} cm) to a galaxy cluster (10^{22} cm).

- $d_i, \rho_i \ll L$ except in laser plasmas and at the boundaries of magnetospheres
- R_m is enormous in astrophysics: ideal MHD, except in thin sheets (module 15)
- λ_{mfp} often exceeds L (tokamak, magnetotail, solar wind): MHD without collisions?

The MHD limit

The MHD ordering

Assume

- lengths $L \gg d_i, \rho_i$ and times $\gg 1/\Omega_i$: drop Hall, pressure and inertia terms
- $U \ll c$: drop the displacement current, $\nabla \times \boldsymbol{B} = 4\pi \boldsymbol{j}/c$, and the force $\rho_c E$ of the charge density
- quasi-neutrality, $n_e = Z n_i$
- a closure for p: adiabatic, $\mathrm{d}(p\rho^{-\gamma})/\mathrm{d}t = 0$

Ohm's law becomes $\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B}/c = \boldsymbol{j}/\sigma$: resistive MHD; with $\boldsymbol{j}/\sigma \to 0$, ideal MHD.

The induction equation

Faraday, $\partial_t \boldsymbol{B} = -c\nabla \times \boldsymbol{E}$, with $\boldsymbol{E} = -\boldsymbol{U} \times \boldsymbol{B}/c + \boldsymbol{j}/\sigma$ and $\boldsymbol{j} = (c/4\pi)\nabla \times \boldsymbol{B}$:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) - \nabla \times \big(\eta \nabla \times \boldsymbol{B}\big), \qquad \eta = \frac{c^2}{4\pi\sigma}$$

For uniform η: $\nabla \times \nabla \times \boldsymbol{B} = \nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B} = -\nabla^2 \boldsymbol{B}$.

Key result

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B} \qquad (\eta \text{ uniform}, \ \nabla \cdot \boldsymbol{B} = 0)$$

the **induction equation**: advection of the field by the flow against diffusion. $R_m = UL/\eta$ measures which wins.

The MHD equations

Key result

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{U}) = 0$$
$$\rho \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = -\nabla p + \frac{1}{4\pi}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\rho^\gamma}\right) = 0$$
$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}, \qquad \nabla \cdot \boldsymbol{B} = 0$$

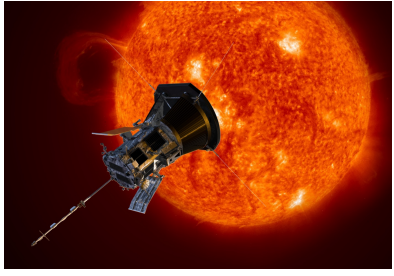
Eight unknowns (ρ, $\boldsymbol{U}$, p, $\boldsymbol{B}$), eight equations; $\boldsymbol{E}$ and $\boldsymbol{j}$ follow from $\boldsymbol{U}$ and $\boldsymbol{B}$. $\boldsymbol{E}$ is not set by charges any more: it is whatever keeps $\boldsymbol{E}' = \boldsymbol{j}/\sigma$ in the fluid frame ($\boldsymbol{E}' = 0$ if ideal). Modules 12–21 use these equations.

Why MHD works without collisions

A fluid needs the particles of a fluid element to stay together.

- **across $\boldsymbol{B}$**: the gyration does it. Particles are tied to field lines within $\rho_i \ll L$: the perpendicular dynamics is fluid-like, collisions or not
- **along $\boldsymbol{B}$**: nothing ties them. Parallel heat flux, pressure anisotropy and Landau damping need kinetic theory (closures, 9a)

Test in the solar wind. The turbulent magnetic spectrum follows MHD down to the ion scale, and steepens at $k \sim 1/\max(d_i, \rho_i)$ (Chen et al. 2014), measured by Wind and by Parker Solar Probe, now within 10 solar radii of the Sun.



NASA/Johns Hopkins APL/Steve Gribben

Hall MHD

Hall MHD: the field follows the electrons

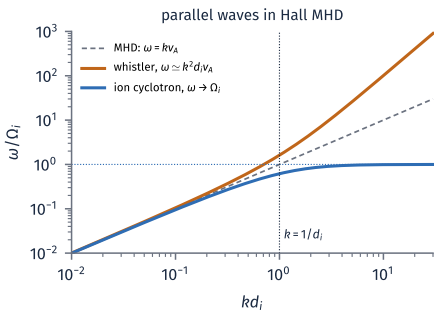
Keep the Hall term, $R_m \to \infty$, p_e uniform:

$$\boldsymbol{E} + \frac{\boldsymbol{U}\times\boldsymbol{B}}{c} = \frac{\boldsymbol{j}\times\boldsymbol{B}}{nec} \;\Leftrightarrow\; \boldsymbol{E} + \frac{\boldsymbol{u}_e\times\boldsymbol{B}}{c} = 0$$

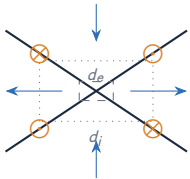
$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}_e\times\boldsymbol{B}), \qquad \boldsymbol{u}_e = \boldsymbol{U} - \frac{c\,\nabla\times\boldsymbol{B}}{4\pi ne}$$

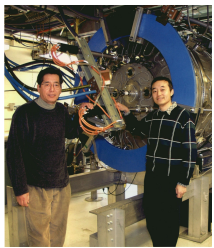
Frozen into the electrons, not the ions. Below d_i the ions decouple; the field moves with the electron fluid.

Parallel waves: $\omega = k v_A\left[\sqrt{1+(k d_i/2)^2} \pm k d_i/2\right]$: the Alfvén wave splits into a whistler and an ion-cyclotron wave.



Hall reconnection: in space and in the laboratory





U.S. DOE / PPPL: M. Yamada, H. Ji at MRX

Within d_i of the X-point the ions decouple, the electrons do not: their in-plane current makes an out-of-plane quadrupolar field ($\odot$, $\otimes$), the signature of Hall reconnection.

- **Wind**, in the magnetotail: the quadrupole detected in situ (Øieroset et al. 2001)
- **MRX** (Princeton): the Hall field measured in the laboratory (Ren et al. 2005)
- **MMS**, four spacecraft 10 km apart: the electron diffusion region itself, at $d_e \sim 1$ km (Burch et al. 2016)

Hall reconnection is fast, independent of η: the topic of module 15.

The Biermann battery

A magnetic field from nothing

Start from $\boldsymbol{B} = 0$. Ohm's law with the pressure term, $\boldsymbol{E} = -\boldsymbol{U} \times \boldsymbol{B}/c - \nabla p_e/(ne)$; Faraday:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) + \frac{c}{e} \nabla \times \left(\frac{\nabla p_e}{n} \right)$$

With $p_e = n T_e$: $\dfrac{\nabla p_e}{n} = \nabla T_e + T_e \nabla \ln n$, and $\nabla \times \nabla T_e = 0$:

$$\nabla \times \left(\frac{\nabla p_e}{n} \right) = \nabla T_e \times \nabla \ln n$$

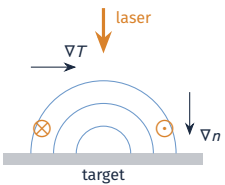
Key result

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) - \frac{c}{ne} \nabla n \times \nabla T_e$$

A source independent of $\boldsymbol{B}$: where $\nabla n \nparallel \nabla T_e$ a field grows from zero (Biermann 1950).

$\nabla p_e/n$ pushes the electrons; if it is not a gradient, it drives an EMF around a loop.

Batteries in the laboratory and in the universe



∇n normal to the target, ∇T radial: a ring of $\boldsymbol{B}$ around the spot

Estimate: $B \sim cT_e t/(eL^2)$.

- **laser plasma**: $T_e = 1$ keV, $L = 100\ \mu$m, $t = 1$ ns: $B \sim 10^6$ G. Measured since Stamper et al. (1971); at **OMEGA** (Rochester), Biermann seeds amplified by turbulence: a dynamo in the laboratory (Tzeferacos et al. 2018)
- **protogalaxies and ionisation fronts**: $T \sim 1$ eV, $L \sim$ kpc, $t \sim 10^8$ yr: $B \sim 10^{-20}$ G, the order of the seeds in simulations of structure formation (Kulsrud et al. 1997)

Key result

Tiny, but not zero: a seed that a dynamo can amplify by many e-folds (modules 16–21).

Which Ohm's law where?

Ohm's law	valid for	examples
ideal, $\boldsymbol{E} = -\boldsymbol{U} \times \boldsymbol{B}/c$	$R_m \gg 1$, $L \gg d_i, \rho_i$	stars, ISM, clusters, solar wind
resistive, $+\boldsymbol{j}/\sigma$	$R_m \lesssim 1$; thin sheets	liquid-metal dynamos, current sheets
Hall, $+\boldsymbol{j} \times \boldsymbol{B}/nec$	$L \lesssim d_i$; weak ionisation	magnetopause, protoplanetary disks
battery, $-\nabla p_e/ne$	$\nabla n \times \nabla T_e \neq 0$	laser plasmas, seed fields
electron inertia	$L \lesssim d_e$	electron diffusion region (MMS)

In tokamaks (JET, ITER) ideal MHD sets the global stability limits; the resistive and two-fluid terms decide what happens in thin layers (tearing modes, module 15).

People

Historical note

- Hall (1879): the transverse voltage in a current-carrying conductor in a magnetic field
- Alfvén (1942): magnetohydrodynamic waves and the frozen-in field; Nobel prize 1970
- Biermann (1950): the thermoelectric generation of magnetic fields in stars
- Stamper et al. (1971): spontaneous magnetic fields in laser-produced plasmas (kilogauss; megagauss a few years later)
- Yamada, Ji and the MRX team (from 1995); Burch and the MMS team (from 2015): reconnection, from the laboratory to the magnetosphere

Check your understanding

1. Why does the friction drop out of the single-fluid momentum equation but not out of Ohm's law?
2. Show that the Hall term is negligible when $L \gg d_i$ for flows at the Alfvén speed.
3. In ideal MHD $\boldsymbol{E}$ is not computed from Poisson's equation. How is it determined?
4. A tokamak has a mean free path of kilometres. Why does MHD describe its global stability?
5. In Hall MHD, which species is the field frozen into, and what changes for waves with $k d_i > 1$?
6. Why does the Biermann battery need ∇n and ∇T_e not parallel?

Exercises in chapter 9 of the notes.

So far, and next

In this module

- one fluid: $\rho\,\mathrm{d}\boldsymbol{U}/\mathrm{d}t = -\nabla p + \boldsymbol{j}\times\boldsymbol{B}/c$; electrons carry $\boldsymbol{j}$
- generalised Ohm's law: resistive, Hall, pressure, inertia; each with its scale (R_m, d_i, ρ_s, d_e)
- MHD: induction equation, $R_m = UL/\eta$; works across $\boldsymbol{B}$ even without collisions
- Hall MHD: field frozen into the electrons; whistlers; Hall reconnection (Wind, MRX, MMS)
- Biermann battery: $\partial_t \boldsymbol{B} \ni -(c/ne)\nabla n \times \nabla T_e$, the seed of cosmic fields

Next: module 10a. Transport: Chapman–Enskog, conductivity, viscosity and heat flux in a magnetised plasma, ambipolar diffusion.