Plasma Physics and Dynamo Theory • Module 9a

From kinetic theory to fluids: moments, closure and two fluids

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Outline

1. Moments of the kinetic equation, with collisions
2. Continuity, momentum, energy
3. The closure problem, and three ways out
4. Testing a closure against kinetic theory
5. The two-fluid plasma: friction, heat exchange, drifts

Reading

Goedbloed & Poedts, ch. 2–3
Kulsrud, ch. 3
Braginskii (1965), §1–2

Notes: chapter 9; ch. 5 (moments)

Where we are

- modules 5–8: the plasma as $f_s(\boldsymbol{x}, \boldsymbol{v}, t)$ in six dimensions. Landau damping, instabilities, collisions, quasi-linear relaxation: all need the velocity dependence
- most of astrophysics (and all of dynamo theory) uses fluids: n, $\boldsymbol{u}$, T in three dimensions

Key result

Today: what a fluid description is, how it follows from f, and what it throws away.
The price: the equations do not close. Something outside them must supply the missing moment.

M05b gave the collisionless continuity and momentum equations. Now: collisions, energy, closure, and two species together.

Moments of the kinetic equation

The moments

For each species s (index dropped), with $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$ the velocity relative to the mean:

$$n = \int f \, \mathrm{d}^3 v, \qquad n\boldsymbol{u} = \int \boldsymbol{v} f \, \mathrm{d}^3 v, \qquad \mathsf{P} = m \int \boldsymbol{w}\boldsymbol{w} \, f \, \mathrm{d}^3 v, \qquad \boldsymbol{q} = \frac{m}{2} \int w^2 \, \boldsymbol{w} \, f \, \mathrm{d}^3 v$$

- scalar pressure and temperature: $p = \frac{1}{3} \mathrm{tr}\, \mathsf{P} = nT$; viscous stress $\boldsymbol{\pi} = \mathsf{P} - p\,\mathsf{I}$
- $\boldsymbol{q}$: heat flux, energy carried by the random motion
- Maxwellian: $\mathsf{P} = nT\,\mathsf{I}$, $\boldsymbol{\pi} = 0$, $\boldsymbol{q} = 0$. Any departure from a Maxwellian shows up as $\boldsymbol{\pi}$, $\boldsymbol{q}$, …

Kinetic equation with collisions (modules 5, 7), $\boldsymbol{a} = (q/m)(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}/c)$:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \boldsymbol{a} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = \sum_b C_{sb}[f, f_b]$$

One recipe for all moments

Multiply by a function $\chi(\boldsymbol{v})$ and integrate over $\boldsymbol{v}$. Write $\langle\chi\rangle = n^{-1}\int \chi f \, \mathrm{d}^3 v$.

- time derivative: $\int \chi \, \partial_t f = \partial_t (n \langle\chi\rangle)$, since χ does not depend on t
- streaming: $\int \chi \, \boldsymbol{v} \cdot \nabla f = \nabla \cdot (n \langle \boldsymbol{v}\chi \rangle)$, since $\boldsymbol{v}$ does not depend on $\boldsymbol{x}$
- force: integrate by parts in $\boldsymbol{v}$; the boundary term vanishes and $\nabla_{\boldsymbol{v}} \cdot \boldsymbol{a} = 0$:

$$\int \chi \, \boldsymbol{a} \cdot \frac{\partial f}{\partial \boldsymbol{v}} \, \mathrm{d}^3 v = -\int f \, \boldsymbol{a} \cdot \frac{\partial \chi}{\partial \boldsymbol{v}} \, \mathrm{d}^3 v = -n \langle \boldsymbol{a} \cdot \nabla_{\boldsymbol{v}} \chi \rangle$$

Key result

$$\frac{\partial}{\partial t}\left(n \langle\chi\rangle\right) + \nabla \cdot \left(n \langle \boldsymbol{v}\chi \rangle\right) - n \langle \boldsymbol{a} \cdot \nabla_{\boldsymbol{v}} \chi \rangle = \sum_b \int \chi \, C_{sb} \, \mathrm{d}^3 v$$

The flux $n \langle \boldsymbol{v}\chi \rangle$ always involves one more power of $\boldsymbol{v}$ than χ: this is where the trouble starts.

$\chi = 1$ and $\chi = mv$: particles and momentum

$\chi = 1$: $\nabla_{\boldsymbol{v}}\chi = 0$, and collisions conserve particles (7a):

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\boldsymbol{u}) = 0$$

$\chi = m\boldsymbol{v}$: $n\langle m\boldsymbol{v}\boldsymbol{v}\rangle = mn\boldsymbol{u}\boldsymbol{u} + \mathrm{P}$ (cross terms vanish: $\langle \boldsymbol{w}\rangle = 0$), and $n\langle \boldsymbol{a}\cdot\nabla_{\boldsymbol{v}}(m\boldsymbol{v})\rangle = mn\langle \boldsymbol{a}\rangle = qn(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}/c)$. Subtract $m\boldsymbol{u}\times$(continuity):

Key result

$$mn\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla\cdot\mathrm{P} + qn\left(\boldsymbol{E} + \frac{\boldsymbol{u}\times\boldsymbol{B}}{c}\right) + \boldsymbol{R}_s, \qquad \boldsymbol{R}_s = \sum_b \int m\boldsymbol{v}\, C_{sb}\, \mathrm{d}^3 v$$

$\boldsymbol{R}_s$: the friction force per unit volume from the other species. From the Landau operator (7a, 8a): $\boldsymbol{R}_{ss} = 0$ (like particles cannot push their own fluid) and $\boldsymbol{R}_{ei} = -\boldsymbol{R}_{ie}$ (action and reaction).

$\chi = \frac{1}{2}mv^2$: energy (1)

Force term: $\nabla_{\boldsymbol{v}} \frac{1}{2}mv^2 = m\boldsymbol{v}$, and $\boldsymbol{v} \cdot (\boldsymbol{v} \times \boldsymbol{B}) = 0$: the magnetic field does no work, $n \langle \boldsymbol{a} \cdot m\boldsymbol{v} \rangle = qn\, \boldsymbol{u} \cdot \boldsymbol{E}$.

Flux: write $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$ and expand $\frac{1}{2}m(u^2 + 2\boldsymbol{u} \cdot \boldsymbol{w} + w^2)(\boldsymbol{u} + \boldsymbol{w})$; terms linear in $\boldsymbol{w}$ average to zero:

$$n \left\langle \frac{1}{2}mv^2 \boldsymbol{v} \right\rangle = \frac{1}{2}mnu^2 \boldsymbol{u} + \frac{3}{2}p\, \boldsymbol{u} + \mathsf{P} \cdot \boldsymbol{u} + \boldsymbol{q}$$

Key result

$$\frac{\partial}{\partial t}\left(\frac{1}{2}mnu^2 + \frac{3}{2}p\right) + \nabla \cdot \left(\frac{1}{2}mnu^2 \boldsymbol{u} + \frac{3}{2}p\, \boldsymbol{u} + \mathsf{P} \cdot \boldsymbol{u} + \boldsymbol{q}\right) = qn\, \boldsymbol{u} \cdot \boldsymbol{E} + \sum_b \int \frac{1}{2}mv^2 C_{sb}\, \mathrm{d}^3 v$$

Energy flux: kinetic energy carried along, enthalpy-like terms, work of the pressure, and $\boldsymbol{q}$, a new unknown.

$\chi = \frac{1}{2}mv^2$: energy (2), the temperature equation

Subtract $\boldsymbol{u}\cdot$(momentum equation), which removes the kinetic energy of the mean flow, and use continuity. With $\mathrm{d}/\mathrm{d}t = \partial_t + \boldsymbol{u}\cdot\nabla$:

Key result

$$\frac{3}{2}n\frac{\mathrm{d}T}{\mathrm{d}t} + p\,\nabla\cdot\boldsymbol{u} = -\nabla\cdot\boldsymbol{q} - \boldsymbol{\pi} : \nabla\boldsymbol{u} + Q_s$$

- $p\,\nabla\cdot\boldsymbol{u}$: compression heats, expansion cools
- $\nabla\cdot\boldsymbol{q}$: heat conduction; $\boldsymbol{\pi} : \nabla\boldsymbol{u}$: viscous heating
- Q_s: heat given to species s by collisions, in its own rest frame ($\sum_b W_{sb} - \boldsymbol{u}_s\cdot\boldsymbol{R}_s$)

Check: with $\boldsymbol{q} = 0$, $\boldsymbol{\pi} = 0$, $Q = 0$, and $\mathrm{d}n/\mathrm{d}t = -n\nabla\cdot\boldsymbol{u}$,

$$\frac{3}{2}n\frac{\mathrm{d}T}{\mathrm{d}t} = T\frac{\mathrm{d}n}{\mathrm{d}t} \;\Rightarrow\; T \propto n^{2/3}, \quad p \propto n^{5/3} : \text{ the adiabatic law of a monatomic gas}$$

The hierarchy never closes

equation for	contains	needs
n (continuity)	$\boldsymbol{u}$	momentum equation
$\boldsymbol{u}$ (momentum)	P, $\boldsymbol{R}$	pressure equation, collision moments
P, T (energy)	$\boldsymbol{q}$, $\boldsymbol{\pi}$, Q	third moments
$\boldsymbol{q}$	fourth moments	…

- streaming $\boldsymbol{v} \cdot \nabla f$ always raises the order by one: the same structure as BBGKY (5a), now in powers of $\boldsymbol{v}$
- the collision moments $\boldsymbol{R}$, Q depend on the whole f, not on n, $\boldsymbol{u}$, T alone

Key result

A closure expresses the highest moment kept in terms of lower ones. It is *physics*, not mathematics: it says which processes are fast enough to fix the shape of f.

Closures

Three ways out

1. **Cold plasma**: P = 0. Valid when $\omega/k \gg v_t$ and thermal effects are not the point (waves, M11)
2. **Equation of state** $p \propto n^\gamma$ ($\boldsymbol{q} = 0$, $\boldsymbol{\pi} = 0$):
   - isothermal, $\gamma = 1$: heat flows fast enough to keep T uniform
   - adiabatic: no heat flows. If only N of the 3 directions are compressed (and no collisions share the energy), $\frac{N}{2}\, n\, \mathrm{d}T/\mathrm{d}t = T\, \mathrm{d}n/\mathrm{d}t$, so

$$\gamma = \frac{N+2}{N}: \qquad N = 3 \to \tfrac{5}{3}, \quad N = 1 \to 3$$

3. **Collisional closure**: if $\lambda_{\mathrm{mfp}} \ll L$, f is close to a local Maxwellian, and $\boldsymbol{q}$, $\boldsymbol{\pi}$, $\boldsymbol{R}$ follow from the small correction (Chapman–Enskog, module 10a): $\boldsymbol{q} = -\kappa \nabla T$, …

Magnetised and collisionless: two pressures, $p_\parallel$ and $p_\perp$, and the double-adiabatic laws $p_\perp/(nB) = \text{const}$, $p_\parallel B^2/n^3 = \text{const}$ (Chew, Goldberger & Low 1956), from μ and J conservation (4a).

Isothermal or adiabatic? Compare the wave with the particles

A wave of frequency ω and wavenumber k compresses the gas. The particles that carry heat move at $\sim v_t$.

$\omega/k \gg v_t$: in one period particles cross only a fraction of a wavelength. No heat leaves the compressed region: **adiabatic**. Compression is along $\boldsymbol{k}$ only: $N = 1$, $\gamma = 3$.

$\omega/k \ll v_t$: particles run across many wavelengths per period and even out T: **isothermal**, $\gamma = 1$.

$\omega/k \sim v_t$: neither. Many particles are resonant, and the fluid misses what they do (Landau damping).

Test 1: Langmuir waves from the fluid equations

Electrons, ions fixed, 1D; equilibrium n_0, $\boldsymbol{u} = 0$, $E = 0$; $p = p(n)$ with $\mathrm{d}p/\mathrm{d}n = \gamma T$. Perturbations $\propto e^{i(kx-\omega t)}$:

$$-i\omega\, n_1 + ikn_0 u_1 = 0, \qquad -i\omega\, mn_0 u_1 = -en_0 E_1 - ik\gamma T\, n_1, \qquad ikE_1 = -4\pi e\, n_1$$

Eliminate $u_1 = \omega n_1/(kn_0)$ and $E_1 = 4\pi i e n_1/k$:

$$-i\frac{\omega^2}{k}\, mn_1 = -\frac{4\pi i e^2 n_0}{k}\, n_1 - ik\gamma T\, n_1$$

Key result

$$\omega^2 = \omega_p^2 + \gamma\, k^2 v_t^2, \qquad v_t^2 = T/m$$

Kinetic theory (6a): $\omega^2 = \omega_p^2 + 3k^2 v_t^2$. The fluid agrees only with $\gamma = 3$: one-dimensional adiabatic.

Test 2: ion-acoustic waves

Now $v_{ti} \ll \omega/k \ll v_{te}$ (6b). Each species gets the closure its speed asks for:

- electrons: isothermal and inertia-free, $0 = -T_e \nabla n_e + e n_e \nabla \phi$, so $n_e = n_0 e^{e\phi/T_e}$ (Boltzmann)
- ions: adiabatic, $\gamma_i = 3$

Ion momentum and continuity, quasi-neutral $n_{i1} = n_{e1} = n_0 e\phi_1/T_e$ (long waves):

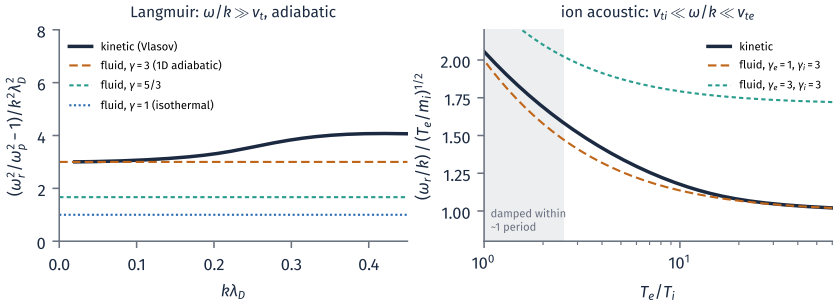
$$-i\omega m_i n_0 u_{i1} = -ik(T_e + 3T_i)\, n_{i1}, \qquad \omega n_{i1} = k n_0 u_{i1}$$

Key result

$$\frac{\omega^2}{k^2} = \frac{T_e + 3T_i}{m_i} \qquad \text{(the kinetic result of 6b, for } T_e \gg T_i\text{)}$$

Adiabatic electrons ($\gamma_e = 3$) would give $3T_e$: wrong by $\sqrt{3}$ in the sound speed.

The test, numerically



Left: from the exact Vlasov root, $(\omega_r^2/\omega_p^2 - 1)/k^2\lambda_D^2 \to 3$ as $k \to 0$. Right: hydrogen, $k\lambda_{De} = 0.1$; $\gamma_e = 1$, $\gamma_i = 3$ tracks the kinetic phase speed. Shaded: damped within about a period, no fluid wave.

A closure can get the real frequency right. It cannot give Landau damping: a fluid has no resonant particles.

Closures in the solar wind

Does the Sun have a wind? Parker (1958): a hot corona expands into a supersonic wind. For a polytropic corona $p \propto n^\gamma$, a wind that starts subsonic and becomes supersonic exists only for $\gamma < 3/2$ (notes, exercise). With $\gamma = 5/3$ there is no such wind. Parker took the corona isothermal, because electron heat conduction is very efficient: the closure decides the answer.

Anisotropy at 1 au. Double-adiabatic (CGL) expansion from the Sun predicts $T_\perp \ll T_\parallel$ for the protons. Measured: $T_\perp / T_\parallel$ stays within the bounds set by kinetic instabilities (firehose, mirror) (Hellinger et al. 2006; Bale et al. 2009). Microinstabilities act as an effective collision: the closure there is set by kinetic physics.

The two-fluid plasma

The two-fluid equations

For $s = e, i$, closed by a choice for P_s (or $\boldsymbol{q}_s$, $\boldsymbol{\pi}_s$), $\boldsymbol{R}_s$, Q_s:

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \boldsymbol{u}_s) = 0$$

$$m_s n_s \frac{\mathrm{d}_s \boldsymbol{u}_s}{\mathrm{d}t} = -\nabla \cdot \mathrm{P}_s + q_s n_s \left(\boldsymbol{E} + \frac{\boldsymbol{u}_s \times \boldsymbol{B}}{c} \right) + \boldsymbol{R}_s$$

$$\tfrac{3}{2} n_s \frac{\mathrm{d}_s T_s}{\mathrm{d}t} + p_s \nabla \cdot \boldsymbol{u}_s = -\nabla \cdot \boldsymbol{q}_s - \boldsymbol{\pi}_s : \nabla \boldsymbol{u}_s + Q_s$$

with Maxwell's equations, the sources being the fluids:

$$\nabla \cdot \boldsymbol{E} = 4\pi \sum_s q_s n_s, \qquad \nabla \times \boldsymbol{B} = \frac{4\pi}{c} \sum_s q_s n_s \boldsymbol{u}_s + \frac{1}{c} \frac{\partial \boldsymbol{E}}{\partial t}, \qquad \nabla \times \boldsymbol{E} = -\frac{1}{c} \frac{\partial \boldsymbol{B}}{\partial t}$$

Unknowns: n_s, $\boldsymbol{u}_s$, T_s (2×5) and $\boldsymbol{E}$, $\boldsymbol{B}$: sixteen fields in 3D, instead of two functions in 6D.

Friction and heat exchange

Simplest model: electrons drifting through ions at $\boldsymbol{u}_e - \boldsymbol{u}_i$ lose momentum at the rate ν_{ei}. With $\boldsymbol{j} = -en(\boldsymbol{u}_e - \boldsymbol{u}_i)$ ($Z = 1$):

$$\boldsymbol{R}_{ei} = -m_e n \nu_{ei}(\boldsymbol{u}_e - \boldsymbol{u}_i) = \frac{m_e \nu_{ei}}{e}\boldsymbol{j} = \frac{ne}{\sigma}\boldsymbol{j}, \qquad \sigma = \frac{ne^2}{m_e \nu_{ei}}$$

The kinetic calculation (7b) adds two things: a factor 0.51 in the parallel resistivity, and a thermal force $-0.71\, n\nabla_\parallel T_e$ (fast electrons collide less).

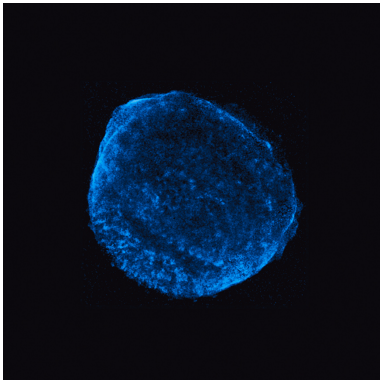
Heat exchange: an electron gives an ion only a fraction $\sim m_e/m_i$ of its energy per collision:

$$Q_i = -Q_e + (\text{friction heating}), \qquad Q_i = 3\frac{m_e}{m_i}\, n\nu_{ei}(T_e - T_i)$$

Key result

T_e and T_i equalise $m_i/m_e \approx 1800$ times more slowly than electrons collide: two-temperature plasmas are common.

Two temperatures in nature



NASA/CXC/Rutgers/G. Cassam-Chenaï, J. Hughes et al.

- **supernova shocks**: a shock heats each species in proportion to its mass, $T_s \propto m_s v_{\mathrm{sh}}^2$. Behind fast shocks the widths of the Balmer lines show $T_e \ll T_p$, as in SN 1006
- **hot accretion flows** (Sgr A*, M87*): ions near the virial temperature, electrons much cooler because they radiate and couple weakly; the images of the Event Horizon Telescope are modelled with T_i/T_e as a free parameter
- **solar wind**: $T_e \neq T_p$ at 1 au; the two fluids are heated differently

The velocity of a magnetised fluid

Slow ($\omega \ll \Omega_s$), weak collisions ($\nu \ll \Omega_s$), isotropic p_s: drop inertia and friction,

$$0 = -\nabla p_s + q_s n_s \left(\boldsymbol{E} + \frac{\boldsymbol{u}_s \times \boldsymbol{B}}{c} \right)$$

Take $\times \boldsymbol{B}$, using $(\boldsymbol{u} \times \boldsymbol{B}) \times \boldsymbol{B} = -B^2 \boldsymbol{u}_\perp$:

$$0 = -\nabla p_s \times \boldsymbol{B} + q_s n_s \, \boldsymbol{E} \times \boldsymbol{B} - \frac{q_s n_s}{c} B^2 \boldsymbol{u}_{s\perp}$$

Key result

$$\boldsymbol{u}_{s\perp} = \underbrace{c \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2}}_{\text{same for both species}} + \underbrace{c \frac{\boldsymbol{B} \times \nabla p_s}{q_s n_s B^2}}_{\text{diamagnetic drift } \boldsymbol{u}_{*s}}$$

$u_* \sim v_t \rho / L$; neglected terms (polarisation, friction) are smaller by ω/Ω, ν/Ω.

The diamagnetic drift: a fluid velocity without a particle drift

- uniform $\boldsymbol{B}$, gradient of n: no guiding centre moves (3b), yet the fluid flows at $\boldsymbol{u}_{*s}$
- the flow is the uneven sum of gyration loops: the magnetisation current of 3b
- opposite for ions and electrons, so it carries a current that does not depend on the charge:

Key result

$$\boldsymbol{j}_\perp = \sum_s q_s n_s \boldsymbol{u}_{*s} = c\,\frac{\boldsymbol{B}\times\nabla p}{B^2}, \qquad \frac{\boldsymbol{j}\times\boldsymbol{B}}{c} = \nabla_\perp p$$

the force balance of a magnetised plasma in equilibrium (modules 12–13)

Measured: the **ring current** (4b) lowers B at the ground (D_{st}); in tokamaks a **diamagnetic loop** measures the drop of the toroidal flux, hence the plasma energy.

When do two fluids behave as one?

Add the two momentum equations: the friction cancels ($\boldsymbol{R}_{ei} + \boldsymbol{R}_{ie} = 0$), and so does the electric force if $n_e = Z n_i$. The difference between them is an equation for $\boldsymbol{j}$: Ohm's law.

The fluids separate where electrons and ions respond differently:

- lengths below the ion inertial length $d_i = c/\omega_{pi}$ or the ion gyroradius ρ_i
- frequencies near Ω_i
- gradients of p_e: the diamagnetic drift of the electrons

Key result

Next module: the generalised Ohm's law (resistive, Hall, pressure and inertia terms), the ordering that leads to MHD, and a battery that can create magnetic fields from nothing.

People

Historical note

- Bohm & Gross (1949): the dispersion relation of Langmuir waves, with its $3k^2v_t^2$
- Chew, Goldberger & Low (1956): the double-adiabatic equations of a collisionless magnetised plasma
- Parker (1958): the solar wind, from an isothermal corona
- Braginskii (1958, 1965): the transport equations of a magnetised two-fluid plasma
- Hammett & Perkins (1990): “Landau-fluid” closures, which put an approximation of Landau damping into fluid equations

Check your understanding

1. Why does the magnetic field drop out of the energy equation, but not out of the momentum equation?
2. Which unknown appears in the energy equation, and why does every moment equation contain a new one?
3. Langmuir waves need $\gamma = 3$; the electrons in an ion-acoustic wave need $\gamma_e = 1$. Why?
4. What does a fluid closure miss for $T_e \simeq T_i$ ion-acoustic waves?
5. No guiding centre drifts in a uniform field, yet the fluid flows at $\boldsymbol{u}_*$. Who carries the current?
6. Why can ions and electrons in a hot plasma keep different temperatures for a long time?

Exercises in chapter 9 of the notes.

So far, and next

In this module

- moments of the kinetic equation: continuity, momentum (with friction), energy
- every equation needs the next moment: a closure is needed
- closures: cold, isothermal, adiabatic ($\gamma = (N+2)/N$), collisional, CGL
- choose by comparing ω/k with v_t; fluids miss Landau damping
- two fluids: friction $\boldsymbol{R}_{ei} = (ne/\sigma)\boldsymbol{j}$, slow T_e–T_i exchange, diamagnetic drift

Next: module 9b. The generalised Ohm's law, the ordering to MHD, Hall MHD and the Biermann battery.