Plasma Physics and Dynamo Theory • Module 8a

Nonlinear wave–particle interaction: quasi-linear theory

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Outline

0. From module 7a: what the Landau operator does
1. One wave or many waves
2. The quasi-linear equations
3. Waves and particles together: the plateau
4. When it works: a numerical test
5. Beams in the solar wind: waves against collisions; stochastic acceleration

Reading

Krall & Trivelpiece, ch. 10
Nicholson, ch. 8
Helander & Sigmar, ch. 3–4

Notes: chapter 8; ch. 7, §7.3–7.5

What the Landau operator does

The Landau operator (1): relaxation

Module 7a: $C_{ab} = \partial_{v_i} J_i^{ab}$, a flux in velocity space, symmetric under $a \leftrightarrow b$, $\boldsymbol{v} \leftrightarrow \boldsymbol{v}'$.
Consequences (proofs: notes, §7.3):

- **conservation**: number, total momentum and total energy
- H**-theorem**: $H = \sum_a \int f_a \ln f_a \, \mathrm{d}^3 v$ never increases,

$$\left(\frac{\partial H}{\partial t}\right)_{\mathrm{coll}} = -\frac{1}{2} \sum_{a,b} K_{ab} \iint f_a f_b' \frac{|\boldsymbol{w}_\perp|^2}{|\boldsymbol{v} - \boldsymbol{v}'|} \, \mathrm{d}^3 v \, \mathrm{d}^3 v' \leq 0, \qquad \boldsymbol{w} = \frac{\partial_{\boldsymbol{v}} \ln f_a}{m_a} - \frac{\partial_{\boldsymbol{v}'} \ln f_b'}{m_b}$$

  ($\perp$: the part of $\boldsymbol{w}$ across $\boldsymbol{v} - \boldsymbol{v}'$)
- **equilibrium**: $C = 0$ only for Maxwellians with a common T and mean velocity (7a: these give $C = 0$; nothing else does)

Key result

Everything relaxes to a Maxwellian: T_e and T_i equalise, currents decay (resistivity), the fine structure left by Landau damping is smoothed away.

The Landau operator (2): conductivity and runaway

Spitzer conductivity. A current is a drift of the electrons; fast electrons collide less ($\tau \propto v^3$) and carry most of it. From the linearised Landau equation (notes, §7.4), with $\sigma_0 = n_e e^2/m_e \nu_{ei}$ of ch. 2:

$$\sigma_L = \frac{32}{3\pi}\,\sigma_0 \approx 3.4\,\sigma_0 \text{ (ions only)}, \qquad \sigma_{\parallel} = 1.96\,\sigma_0 \text{ (with e–e collisions; Spitzer \& Härm 1953)}$$

Runaway electrons. The friction on a fast electron *decreases* with speed:

$$F_s = \frac{4\pi n_e e^4 \ln\Lambda}{m_e v^2} < eE \quad \text{for} \quad v > v_c = v_{te}\sqrt{E_D/E}, \qquad E_D = \frac{4\pi n_e e^3 \ln\Lambda}{T_e}$$

Above v_c, acceleration without limit (notes, §7.5); relativity: no runaway below $E_c = (T_e/m_e c^2)\,E_D$.

Key result

Tokamak disruptions, gamma-ray flashes, hard X-rays in flares. Derivations: module 7b.

Where we are

Key result: recap

- ch. 6: a Langmuir wave grows or damps at $\gamma_k = \dfrac{\pi}{2}\dfrac{\omega_p^3}{nk^2} f_0'\left(\dfrac{\omega_p}{k}\right)$; a bump with $f_0' > 0$ is unstable
- ch. 6: a single large wave traps electrons, bounce frequency $\omega_b = (ekE_0/m)^{1/2}$
- ch. 7: random kicks = diffusion in velocity, $\dfrac{\partial f}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2}{\partial v^2}(Bf)$, $B = \left\langle \Delta v^2 \right\rangle / \Delta t$

Linear theory keeps f_0 fixed, so an unstable wave grows as $e^{\gamma t}$ for ever. **Today**: let f_0 respond to the waves. With many waves, the resonant electrons diffuse in velocity, and the bump flattens into a plateau.

One wave or many waves

What stops the growth?

Two limits, both nonlinear:

One coherent wave (ch. 6)

- resonant electrons are trapped in the troughs and bounce at ω_b
- after $t \sim 1/\omega_b$ they give the energy back: growth stops
- the wave settles to a stationary nonlinear wave, a BGK mode (next slide)

A broad spectrum, random phases (today)

- each electron feels many waves, one after the other
- the kicks are uncorrelated: a random walk in v
- the average $f_0(v, t)$ diffuses where the waves are

An unstable bump excites all k with ω_p/k on its rising side: the second case is the generic one.

Quasi-linear theory: linear waves; the only nonlinearity kept is the slow change of f_0.

One wave: trapping and BGK modes

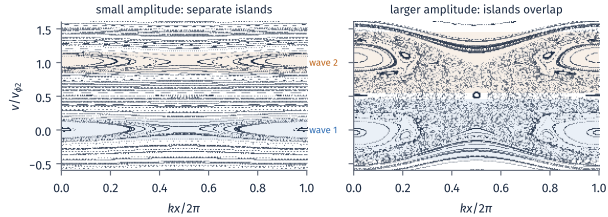
In the wave frame ($v_\phi = \omega/k$) the potential $\phi_0 \cos kx$ is static, and $W = \frac{1}{2}m(v - v_\phi)^2 - e\phi(x)$ is conserved.

- $|v - v_\phi| < 2\omega_b/k$: trapped, closed orbits (the "cat's eyes"), bounce frequency $\omega_b = \sqrt{ekE_0/m}$
- first half bounce they take energy from the wave, second half they give it back: the amplitude oscillates
- orbits bounce at different rates and dephase: f flattens inside the eye, $\partial_v f = 0$ at v_ϕ, the exchange stops

Key result

BGK modes (Bernstein, Greene & Kruskal 1957): any $f(W)$ solves the stationary Vlasov equation; Poisson then fixes ϕ. Exact nonlinear waves of almost any shape exist, with matching trapped electrons.

Many waves: resonances overlap



Electrons in two equal waves, phase velocities 0 and $v_{\phi 2}$, plotted only once per period of wave 2 (a stroboscopic map): a **curve** is a regular orbit, a **cloud of dots** is a chaotic one. Each wave traps a band of half-width $2\omega_b/k$ (shaded).

- bands apart: regular orbits, trapping in each wave separately
- bands overlap: orbits wander between them: stochastic motion (Chirikov)
- many waves, overlapping bands: the velocity performs a random walk

$\ddot{x} = -A[\sin x + \sin(x - v_{\phi 2} t)]$, $k = 1$, $A = \omega_b^2 = 0.006,\ 0.05$; 90 electrons, 300 periods, leapfrog.

The quasi-linear equations

Slow and fast parts

Electrons ($-e$), ions fixed, one dimension:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}E\,\frac{\partial f}{\partial v} = 0, \qquad \frac{\partial E}{\partial x} = -4\pi e\left(\int f\,\mathrm{d}v - n\right)$$

Split $f = f_0(v,t) + f_1(x,v,t)$, with $f_0 = \langle f \rangle$ the average over a length $L \gg$ all wavelengths: $\langle f_1 \rangle = 0$, $\langle E \rangle = 0$.

Average the Vlasov equation. $\langle v\,\partial_x f \rangle = v\,\langle \partial_x f_1 \rangle = 0$ (periodic), and $\langle E \rangle\,\partial_v f_0 = 0$:

$$\frac{\partial f_0}{\partial t} = \frac{e}{m}\left\langle E\,\frac{\partial f_1}{\partial v}\right\rangle = \frac{e}{m}\,\frac{\partial}{\partial v}\,\langle E f_1 \rangle$$

(E does not depend on v.) The background changes only through the correlation of E and f_1: second order in the amplitude, so it is slow.

The fluctuations

Subtract the average from the Vlasov equation:

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} - \frac{e}{m}E\frac{\partial f_0}{\partial v} = \frac{e}{m}\left(E\frac{\partial f_1}{\partial v} - \left\langle E\frac{\partial f_1}{\partial v}\right\rangle\right)$$

Quasi-linear approximation: drop the right-hand side (wave–wave coupling). Then f_1 obeys the *linear* equation of ch. 6, with the slowly changing $f_0(v,t)$ in place of the initial one.

Fourier modes, $E = \sum_k E_k\, e^{i(kx-\omega_k t)}$, $\omega_k = \omega_r + i\gamma_k$:

$$-i(\omega_k - kv)f_{1k} - \frac{e}{m}E_k\frac{\partial f_0}{\partial v} = 0 \quad\Longrightarrow\quad f_{1k} = \frac{ie}{m}\frac{E_k}{\omega_k - kv}\frac{\partial f_0}{\partial v}$$

- E real: $E_{-k} = E_k^*$ and $\omega_{-k} = -\omega_k^*$ (same growth rate, opposite real frequency)

The correlation $\langle E f_1 \rangle$

In the product of two sums only $k' = -k$ survives the average:

$$\langle E f_1 \rangle = \sum_k E_k^* f_{1k} = \frac{ie}{m} \frac{\partial f_0}{\partial v} \sum_k \frac{|E_k|^2}{\omega_k - kv}$$

Pair the terms k and $-k$, using $\omega_{-k} = -\omega_k^*$:

$$\frac{1}{\omega_k - kv} + \frac{1}{-\omega_k^* + kv} = \frac{\omega_k^* - \omega_k}{|\omega_k - kv|^2} = \frac{-2i\gamma_k}{(\omega_r - kv)^2 + \gamma_k^2}$$

so that $\sum_k |E_k|^2/(\omega_k - kv) = -i \sum_k \gamma_k |E_k|^2/[(\omega_r - kv)^2 + \gamma_k^2]$ (each pair counted once).

Key result: quasi-linear diffusion

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\left(D(v) \frac{\partial f_0}{\partial v} \right), \qquad D(v) = \frac{e^2}{m^2} \sum_k |E_k|^2 \frac{\gamma_k}{(\omega_r - kv)^2 + \gamma_k^2}$$

Resonant electrons

For $|\gamma_k| \ll \omega_r$ the Lorentzian is narrow, of area π:

$$\frac{\gamma_k}{(\omega_r - kv)^2 + \gamma_k^2} \longrightarrow \pi\,\delta(\omega_r - kv)$$

Only electrons with $v = \omega_r/k$, *surfing* the wave, diffuse. The remaining non-resonant part is the reversible sloshing of the bulk in the wave field (notes).

Spectrum: electric energy density $\langle E^2 \rangle / 8\pi = \sum_k |E_k|^2/8\pi = \int W_k \,\mathrm{d}k$; for Langmuir waves $\omega_r \simeq \omega_p$, and $\delta(\omega_p - kv) = \delta(k - \omega_p/v)/|v|$:

Key result

$$D(v) = \frac{8\pi^2 e^2}{m^2} \int W_k\,\delta(\omega_p - kv)\,\mathrm{d}k = \frac{8\pi^2 e^2}{m^2|v|}\,W_{k=\omega_p/v}$$

A diffusion equation, as in ch. 7, but the kicks come from waves, and only resonant electrons receive them.

Why diffusion: an estimate

An electron at v sees the phase $kx - \omega_p t = (kv - \omega_p)t$ of each wave. Waves in a band Δk about resonance stay in phase with it for the correlation time

$$\tau_c \simeq \frac{1}{v\,\Delta k}$$

During τ_c the field of the band pushes coherently: $\Delta v \simeq (e/m) E_{\rm band} \tau_c$, with $E^2_{\rm band} \simeq 8\pi W_k \Delta k$. Then the phases are lost, and the next kick is independent:

$$D \simeq \frac{\langle \Delta v^2 \rangle}{\tau_c} \simeq \frac{e^2}{m^2} E^2_{\rm band}\, \tau_c \simeq \frac{e^2}{m^2} \frac{8\pi W_k \Delta k}{v \Delta k} = \frac{8\pi e^2 W_k}{m^2 v}$$

Key result

Exact result: an extra π. Δk cancels: only the energy per unit k at resonance matters. Needs $\tau_c \ll 1/\omega_b$, otherwise the electron is trapped.

Waves and particles together

A closed system

The waves grow or damp at the linear rate, computed with the *current* f_0:

$$\frac{\partial W_k}{\partial t} = 2\gamma_k W_k, \qquad \gamma_k = \frac{\pi}{2}\,\frac{\omega_p^3}{nk^2}\,\frac{\partial f_0}{\partial v}\bigg|_{v=\omega_p/k}$$

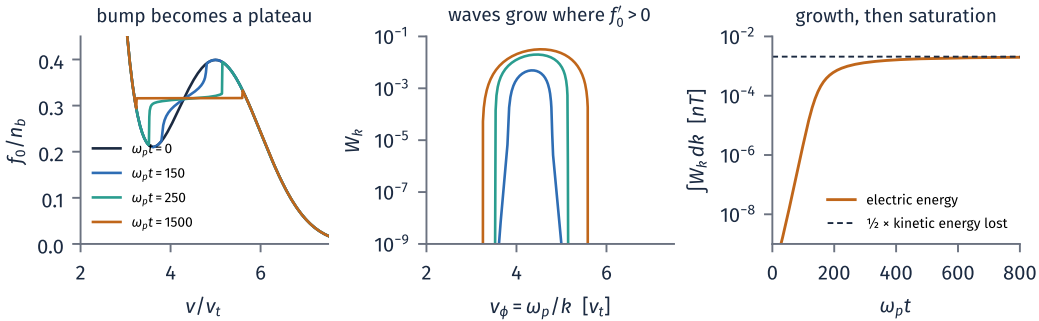
($W \propto |E_k|^2$, hence $2\gamma_k$.)

Key result: quasi-linear equations

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\left(\frac{8\pi^2 e^2}{m^2 v}\, W_{\omega_p/v}\,\frac{\partial f_0}{\partial v}\right), \qquad \frac{\partial W_k}{\partial t} = \frac{\pi\omega_p^3}{nk^2}\,\frac{\partial f_0}{\partial v}\bigg|_{\omega_p/k} W_k$$

- where $f_0' > 0$, waves grow; they make the electrons diffuse, which lowers f_0'
- the evolution stops only when $W_k\ \partial_v f_0 = 0$ everywhere: either no waves, or a flat f_0
- end state: a plateau over the resonant velocities, with a saturated spectrum of waves

The bump becomes a plateau



Quasi-linear equations solved numerically: warm beam, $n_b = 0.01\,n$, $v_b = 5v_t$, $\Delta v_b = v_t$. Waves grow only where $f_0' > 0$, and stop when the plateau forms ($\omega_p t \approx 300$). The electric energy is half the kinetic energy lost by the beam (dashed): why?

Conservation of momentum

Electron momentum, by parts:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int mv\, f_0\,\mathrm{d}v = \int mv\,\frac{\partial}{\partial v}\left(D f_0'\right)\mathrm{d}v = -\int mD\, f_0'\,\mathrm{d}v = -\frac{8\pi^2 e^2}{m}\int \frac{W_{\omega_p/v}}{v}\, f_0'(v)\,\mathrm{d}v$$

Change variable to $k = \omega_p/v$, $\mathrm{d}v = -(\omega_p/k^2)\,\mathrm{d}k$:

$$= -\frac{8\pi^2 e^2}{m}\int \frac{W_k}{k}\, f_0'\left(\frac{\omega_p}{k}\right)\mathrm{d}k$$

Waves: energy $2W_k$ (half electric, half the kinetic energy of the oscillating electrons), momentum $2W_k/v_\phi = 2W_k k/\omega_p$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{2kW_k}{\omega_p}\,\mathrm{d}k = \int \frac{2k}{\omega_p}\,2\gamma_k W_k\,\mathrm{d}k = \frac{8\pi^2 e^2}{m}\int \frac{W_k}{k}\, f_0'\left(\frac{\omega_p}{k}\right)\mathrm{d}k$$

using $\omega_p^2 = 4\pi n e^2/m$. Equal and opposite. The same steps with $\frac{1}{2}mv^2$ give energy conservation.

How much energy goes into the waves?

Combine the two equations: at resonance $D\,\partial_v f_0 = (2\omega_p/mv^3)\,\partial_t W$, so

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\left(\frac{2\omega_p}{mv^3}\,\frac{\partial W}{\partial t}\right) \implies \frac{2\omega_p}{mv^3}\,W^{\text{final}}_{\omega_p/v} = \int_{v_1}^{v} \left[f_0(v',\infty) - f_0(v',0)\right] \mathrm{d}v'$$

(v_1: lower edge of the plateau, initial waves negligible).

Cold beam $n_b\,\delta(v - v_b)$ on a negligible background: plateau n_b/v_b for $0 < v < v_b$ (same particles).

- the integral is $n_b v/v_b$, so $W = mn_b v^4/(2\omega_p v_b)$: waves pile up near the beam speed
- electric energy $\int W\,\mathrm{d}k = \int_0^{v_b} W\,\omega_p\,\mathrm{d}v/v^2 = \frac{1}{6} n_b m v_b^2$; wave energy twice that

Key result

The beam keeps half its momentum and gives two thirds of its energy to the waves, in a time $\sim (n/n_b)\,\omega_p^{-1}$ times a logarithm.

When it works; numerical tests

When is quasi-linear theory valid?

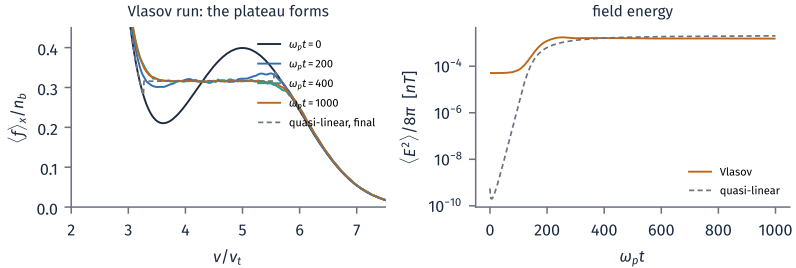
Resonant phase velocities spread over Δv_ϕ, adjacent waves separated by δv_ϕ:

- **overlap**: $\delta v_\phi \ll \omega_b/k$: the trapping bands of neighbouring waves overlap (stochastic motion)
- **short memory**: $\omega_b/k \ll \Delta v_\phi$, i.e. $\omega_b \tau_c \ll 1$: no electron completes a bounce in one wave
- **slow background**: $\gamma_k \ll \omega_p$ and $\gamma_k \tau_c \ll 1$, i.e. a *warm* beam: the kinetic, not the reactive, instability of ch. 6
- **weak turbulence**: $\sum W_k \ll nT$; wave–wave coupling neglected

Key result

A broad spectrum of small waves, many in each trapping width. One coherent wave, or a cold beam, is the opposite limit: trapping and BGK structures (ch. 5–6).

A test: the full Vlasov equation

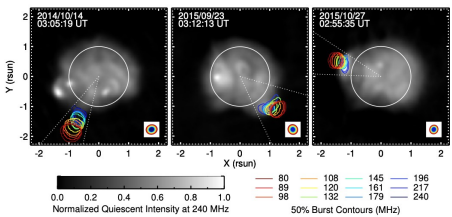


The same beam, now with the full Vlasov–Poisson equations, $L = 1000\,\lambda_D$ (many unstable modes), started from random noise.

- the space average $\langle f \rangle_x$ flattens into the quasi-linear plateau
- the field energy saturates close to the quasi-linear level; it starts from a higher noise level, so it saturates earlier

In nature and in the laboratory

Sturrock's dilemma resolved



P. McCauley, Murchison Widefield Array, CC BY-SA 4.0

- ch. 6: a type III beam should lose its energy within $\sim$ 100 km, yet reaches 1 au
- quasi-linear relaxation is fast, but it produces a plateau, not a stopped beam: electrons move along the plateau and keep about half their momentum
- faster electrons arrive first: at the front waves grow, at the back the slower electrons reabsorb them. The beam carries its waves with it
- in situ (ISEE-3, Wind, Parker Solar Probe, Solar Orbiter): beams with nearly flat, marginally stable distributions, and bursty Langmuir waves

Waves against collisions

The plateau is not a Maxwellian: by the H-theorem, collisions must erase it. How fast?

Collisions: a fast electron slows down at $\nu_s(v) = \frac{1}{v}\left|\frac{\mathrm{d}v}{\mathrm{d}t}\right| = \frac{4\pi n_e e^4 \ln\Lambda}{m_e^2 v^3} \propto v^{-3}$.

Waves: quasi-linear relaxation in $\tau_{\mathrm{QL}} \sim (n/n_b)\,\omega_p^{-1}\times$ a logarithm.

10 keV beam, $v \approx 6\times10^9$ cm/s	n_e (cm^{-3})	$1/\nu_s$	travel time
low corona	10^9	≈ 13 s	~ 10 s (one $R_\odot$)
solar wind at 1 au	5	≈ 65 yr	≈ 40 min

($\ln\Lambda \approx 20$–25; $\omega_p^{-1} \approx 0.6$ ns and 8 μs.)

Key result

In the wind only waves shape the beam: τ_{QL} is seconds to minutes, collisions take decades. Near the Sun collisions compete, and slow the slower beam electrons first.

Stochastic acceleration

A broad spectrum of total energy density W, spread over $\Delta k \sim k \sim \omega_p / v_t$: $W_k \sim W v_t / \omega_p$, and for $v \sim v_t$

$$D \sim \frac{8\pi^2 e^2 W}{m^2 \omega_p}$$

Time to change the speed by Δv (random walk, $\Delta v^2 \simeq 2Dt$), using $e^2 = m\omega_p^2 / 4\pi n$:

$$t_{\mathrm{acc}} \simeq \frac{\Delta v^2}{2D} = \frac{1}{4\pi\omega_p} \frac{nT}{W} \left(\frac{\Delta v}{v_t} \right)^2$$

Key result

$W/nT = 10^{-3}$ and $\Delta v = v_t$: $\omega_p t_{\mathrm{acc}} \approx 80$. A wave energy of a thousandth of the thermal energy doubles the speed of tail electrons in about a hundred plasma periods.

The same physics is the microphysics of second-order Fermi acceleration (ch. 4): turbulence heats tails in solar flares, in the auroral zone, and accelerates cosmic rays.

People

Historical note

- Romanov & Filippov (1961); Vedenov, Velikhov & Sagdeev (1961–62); Drummond & Pines (1962): quasi-linear theory, independently in the Soviet Union and in the United States (the last two at the 1961 Salzburg conference on fusion research)
- Sturrock (1964): the type III dilemma; its quasi-linear resolution occupied the following decades
- Bernstein, Greene & Kruskal (1957): exact nonlinear stationary waves (BGK modes); O'Neil (1965): the trapped-electron oscillation of the amplitude
- Chirikov (1959, 1979): overlap of resonances as the criterion for stochastic motion

Check your understanding

1. Why does the averaged Vlasov equation contain no first-order term in the wave amplitude?
2. Where in the derivation does the resonance $v = \omega_p / k$ come from? Why are non-resonant electrons not diffused?
3. Why is the electric energy of the waves half of the kinetic energy lost by the beam?
4. A cold beam gives two thirds of its energy to the waves. Where does the remaining third go?
5. Why does quasi-linear theory fail for a single large-amplitude wave?
6. The plateau is not a Maxwellian. Why does it survive from the Sun to 1 au?

Exercises in chapter 8 of the notes.

So far, and next

In this module

- many waves with random phases: resonant electrons diffuse in velocity
- $D(v) = (8\pi^2 e^2/m^2 v)\, W_{\omega_p/v}$; waves grow at $2\gamma_k$
- end state: plateau plus saturated waves; momentum and energy conserved
- cold beam: $2/3$ of its energy to the waves
- validity: overlapping, short-lived resonances; one wave gives trapping instead
- collisions (Landau operator) erase the plateau only on $1/\nu_s \propto v^3$: decades at 1 au

Next: module 9a. From kinetic theory to fluids: moments of the kinetic equation, the closure problem, the two-fluid equations.