Plasma Physics and Dynamo Theory • Module 7b

Collisions in kinetic theory: properties, conductivity, runaway

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Outline

1. Conservation of particles, momentum and energy
2. Boltzmann's H-theorem for the Landau operator
3. The Maxwellian as the only equilibrium
4. The Lorentz gas: pitch-angle scattering and conductivity
5. Electron–electron collisions: Spitzer–Härm
6. Runaway electrons: Dreicer, E_c, avalanches

Reading

Helander & Sigmar, ch. 3–4
Goldston & Rutherford, ch. 11–12
Spitzer & Härm (1953)

Notes: chapter 7, §7.3–7.5

Where we are

Key result: recap of module 7a

- small-angle Coulomb encounters = diffusion in velocity space: $C = -\partial_i(A_i f) + \frac{1}{2}\partial_i\partial_j(B_{ij} f)$
- beam of field particles: $A_i = -n\Gamma_{ab}(1 + m_a/m_b)\, u_i/u^3$, $B_{ij} = n\Gamma_{ab} U_{ij}(\boldsymbol{u})$, $U_{ij} = (u^2\delta_{ij} - u_i u_j)/u^3$
- Maxwellian field particles: $\psi(x)$; fast particles are deflected before they are slowed
- Landau operator, zero for Maxwellians with common T

Today: what the operator guarantees (conservation, entropy growth, Maxwellian), and what it gives: the true conductivity of a plasma, and the electrons that escape collisions altogether.

Properties of the collision operator

The Landau operator as a divergence

With $K_{ab} = 2\pi q_a^2 q_b^2 \ln\Lambda$ (symmetric in a, b), the Landau operator of module 7a reads

$$C_{ab} = \frac{\partial J_i^{ab}}{\partial v_i}, \qquad J_i^{ab} = \frac{K_{ab}}{m_a} \int U_{ij}(\boldsymbol{v} - \boldsymbol{v}')\, Q_j \, \mathrm{d}^3 v', \qquad Q_j = \frac{f_b(\boldsymbol{v}')}{m_a} \frac{\partial f_a(\boldsymbol{v})}{\partial v_j} - \frac{f_a(\boldsymbol{v})}{m_b} \frac{\partial f_b(\boldsymbol{v}')}{\partial v'_j}$$

Two facts do all the work:

1. $U_{ij}(\boldsymbol{u})$ is *even*: unchanged under $\boldsymbol{v} \leftrightarrow \boldsymbol{v}'$
2. under the double exchange $a \leftrightarrow b$, $\boldsymbol{v} \leftrightarrow \boldsymbol{v}'$: $Q_j \to -Q_j$

Moments are integrals over *both* velocities; the double exchange only renames the integration variables.

Particles: $\int C_{ab}\, \mathrm{d}^3 v = \oint J^{ab} \cdot \mathrm{d}\boldsymbol{S}_v = 0$. Collisions move particles in velocity space; they do not create them.

Momentum

Momentum given to a by b, per unit volume and time. By parts, with $\partial v_k / \partial v_i = \delta_{ik}$:

$$R_{ab,k} = \int m_a v_k \frac{\partial J_i^{ab}}{\partial v_i} \, \mathrm{d}^3 v = -m_a \int J_k^{ab} \, \mathrm{d}^3 v = -K_{ab} \iint U_{kj}(\boldsymbol{v} - \boldsymbol{v}') \, Q_j \, \mathrm{d}^3 v \, \mathrm{d}^3 v'$$

For R_{ba}: exchange the labels *and* rename $\boldsymbol{v} \leftrightarrow \boldsymbol{v}'$. K and U unchanged, $Q_j \to -Q_j$:

Key result

$$\boldsymbol{R}_{ba} = -\boldsymbol{R}_{ab} \qquad \text{(action and reaction between the two populations)}$$

- like particles: $\boldsymbol{R}_{aa} = -\boldsymbol{R}_{aa} = 0$. Collisions within a species cannot change its momentum
- electron–electron collisions alone cannot produce resistance: we will need this for Spitzer

Energy

Same steps, with $\partial(v^2/2)/\partial v_i = v_i$:

$$W_{ab} = \int \tfrac{1}{2} m_a v^2 \, C_{ab} \, \mathrm{d}^3 v = -K_{ab} \iint v_i \, U_{ij}(\boldsymbol{v} - \boldsymbol{v}') \, Q_j \, \mathrm{d}^3 v \, \mathrm{d}^3 v'$$

Now the exchange turns v_i into v_i' *and* Q_j into $-Q_j$:

$$W_{ab} + W_{ba} = -K_{ab} \iint (v_i - v_i') \, U_{ij} \, Q_j = -K_{ab} \iint u_i U_{ij}(\boldsymbol{u}) \, Q_j = 0$$

Key result

Energy conservation = $u_i U_{ij} = 0$: in each encounter the relative speed is unchanged, diffusion is only *across* $\boldsymbol{u}$.

Energy can still flow between species ($W_{ab} \neq 0$): temperature equilibration, ch. 2.

The H-theorem (1)

Boltzmann's function, $-H$ = entropy density (units of k_B):

$$H = \sum_a \int f_a \ln f_a \, \mathrm{d}^3 v$$

Vlasov terms: $\boldsymbol{v} \cdot \nabla G(f)$ and $\boldsymbol{a} \cdot \partial_{\boldsymbol{v}} G(f)$ are phase-space divergences ($\partial_{\boldsymbol{v}} \cdot \boldsymbol{a} = 0$ for Lorentz): they do not change $\int H \, \mathrm{d}^3 x$. Only collisions do.

$$\left(\frac{\partial H}{\partial t}\right)_{\text{coll}} = \sum_{a,b} \int (1 + \ln f_a) \, C_{ab} \, \mathrm{d}^3 v = -\sum_{a,b} \int \frac{\partial \ln f_a}{\partial v_i} \, J_i^{ab} \, \mathrm{d}^3 v$$

(the 1 drops by particle conservation; then by parts). Write the bracket with logarithmic derivatives:

$$Q_j = f_a(\boldsymbol{v}) f_b(\boldsymbol{v}') \, w_j, \qquad w_j = \frac{1}{m_a} \frac{\partial \ln f_a(\boldsymbol{v})}{\partial v_j} - \frac{1}{m_b} \frac{\partial \ln f_b(\boldsymbol{v}')}{\partial v'_j}$$

The H-theorem (2)

The (a, b) term is $-K_{ab} \iint f_a f_b' \left(m_a^{-1} \partial_i \ln f_a\right) U_{ij}\, w_j$.

The (b, a) term, after the double exchange: $m_a^{-1} \partial_i \ln f_a \to m_b^{-1} \partial_i' \ln f_b'$ and $w_j \to -w_j$. The sum contains w_i itself:

$$-K_{ab} \iint f_a f_b'\; w_i U_{ij} w_j \, \mathrm{d}^3 v \, \mathrm{d}^3 v' \qquad \text{(half of it for } a = b\text{)}$$

The quadratic form is never negative:

$$w_i U_{ij} w_j = \frac{w^2 u^2 - (\boldsymbol{w} \cdot \boldsymbol{u})^2}{u^3} = \frac{|\boldsymbol{w}_\perp|^2}{u} \geq 0$$

Key result: H-theorem

$$\left(\frac{\partial H}{\partial t}\right)_{\text{coll}} \leq 0, \qquad \text{with equality only if } \boldsymbol{w} \parallel \boldsymbol{v} - \boldsymbol{v}' \text{ for every pair}$$

The Maxwellian as the only equilibrium

Let $\boldsymbol{G}_a = m_a^{-1}\,\partial_{\boldsymbol{v}} \ln f_a$. Equality requires $\boldsymbol{G}_a(\boldsymbol{v}) - \boldsymbol{G}_b(\boldsymbol{v}') = \lambda\,(\boldsymbol{v} - \boldsymbol{v}')$ for all $\boldsymbol{v}, \boldsymbol{v}'$.

- $b = a$: differences of $\boldsymbol{G}$ always parallel to differences of $\boldsymbol{v} \Rightarrow \boldsymbol{G}_a = -\alpha_a \boldsymbol{v} + \boldsymbol{\beta}_a$ (notes, exercise):

$$\ln f_a = -\tfrac{1}{2} m_a \alpha_a v^2 + m_a \boldsymbol{\beta}_a \cdot \boldsymbol{v} + \text{const} \;\Rightarrow\; f_a \propto e^{-m_a |\boldsymbol{v} - \boldsymbol{V}_a|^2 / 2T_a}$$

- $b \neq a$: $-\alpha_a \boldsymbol{v} + \alpha_b \boldsymbol{v}' + \boldsymbol{\beta}_a - \boldsymbol{\beta}_b \parallel \boldsymbol{v} - \boldsymbol{v}'$ for all $\boldsymbol{v}, \boldsymbol{v}' \Rightarrow \alpha_a = \alpha_b, \boldsymbol{\beta}_a = \boldsymbol{\beta}_b$

Key result: collisional equilibrium

$C = 0 \Leftrightarrow$ all species Maxwellian with the *same* T and the *same* mean velocity.

- $T_e \neq T_i$, or a current, is not equilibrium: it relaxes (ch. 2 rates; resistivity)
- the irreversibility Vlasov lacks: Landau damping stores energy in fine structure of f (echoes undo it); collisions smooth it away for good

Lorentz gas and Spitzer conductivity

The Lorentz gas

Lorentz (1905), for metals: electrons scattered by heavy fixed centres, no electron–electron collisions. For a plasma: ions at rest, $m_e/m_i \to 0$; good for large Z (e–i scattering Z times e–e). Beam result with $\boldsymbol{v}' = 0$, $\Gamma_{ei} = 4\pi Z^2 e^4 \ln\Lambda / m_e^2$:

$$A_i = -n_i \Gamma_{ei} \frac{v_i}{v^3}, \qquad B_{ij} = n_i \Gamma_{ei} U_{ij}(\boldsymbol{v})$$

Flux form, as for Landau: $C = \partial_i \left[\frac{1}{2} B_{ij} \partial_j f + (\frac{1}{2} \partial_j B_{ij} - A_i) f \right]$. Since $\partial_j U_{ij} = -2 v_i / v^3$:

$$\tfrac{1}{2} \partial_j B_{ij} = -n_i \Gamma_{ei} \frac{v_i}{v^3} = A_i \qquad \Rightarrow \qquad C_{ei}[f] = \frac{n_i \Gamma_{ei}}{2} \frac{\partial}{\partial v_i} \left(U_{ij} \frac{\partial f}{\partial v_j} \right)$$

The friction is all “fake”: it only compensates the variation of B_{ij}. No drift remains.

Pitch-angle scattering

$U_{ij}\,\partial_j f = \frac{1}{v}\left(\partial_i f - \hat{v}_i \hat{v}_j\,\partial_j f\right) = \frac{1}{v}(\nabla_v f)_\perp$: the gradient *on the sphere* $|\boldsymbol{v}| = v$. Its divergence has only angular derivatives:

$$\frac{\partial}{\partial v_i}\left(U_{ij}\frac{\partial f}{\partial v_j}\right) = \frac{1}{v^3}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}\right]$$

Key result: Lorentz operator ($\xi = \cos\theta$, no ϕ dependence)

$$C_{ei}[f] = \nu_{ei}(v)\,\frac{1}{2}\,\frac{\partial}{\partial\xi}\left[(1-\xi^2)\frac{\partial f}{\partial\xi}\right], \qquad \nu_{ei}(v) = \frac{n_i\Gamma_{ei}}{v^3} = \frac{4\pi n_e Z e^4 \ln\Lambda}{m_e^2 v^3}$$

- no $\partial/\partial v$: heavy ions only turn $\boldsymbol{v}$; every isotropic $f(v)$ is an equilibrium
- Legendre: $\partial_\xi[(1-\xi^2)\partial_\xi P_l] = -l(l+1)P_l$. Anisotropy l decays at $l(l+1)\nu_{ei}/2$
- $l = 1$ (current) at $\nu_{ei}(v)$, the momentum-transfer rate of ch. 2; $l = 2$ three times faster

Conductivity of a Lorentz gas (1): the perturbed distribution

Weak uniform $\boldsymbol{E} = E\hat{\boldsymbol{z}}$, steady state: $-\dfrac{eE}{m_e}\dfrac{\partial f}{\partial v_z} = C_{ei}[f]$, $\qquad f = f_M + f_1, \quad f_1 = O(E)$ Left: keep f_M; $\partial f_M/\partial v_z = -(v_z/v_{te}^2) f_M = -(m_e v\xi/T_e) f_M$. Right: $C_{ei}[f_M] = 0$.

$$\frac{eE v\,\xi}{T_e} f_M = \nu_{ei}(v)\,\frac{1}{2}\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial f_1}{\partial \xi}\right]$$

Left side $\propto \xi = P_1$: try $f_1 = \xi F(v)$. With $\frac{1}{2}\partial_\xi[(1-\xi^2)\cdot 1] = -\xi$:

Key result

$$f_1 = -\frac{eE v\,\xi}{T_e\,\nu_{ei}(v)} f_M(v)$$

Electrons shifted against $\boldsymbol{E}$; the shift is largest where ν_{ei} is smallest, i.e. for the fast electrons.

Conductivity of a Lorentz gas (2): who carries the current

With $\mathrm{d}^3v = v^2 \mathrm{d}v\, \mathrm{d}\Omega$, $\int \xi^2 \mathrm{d}\Omega = 4\pi/3$:

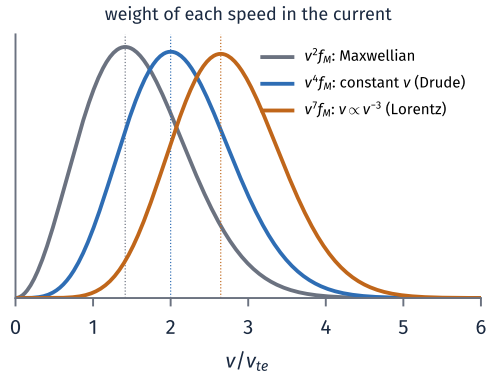
$$j_z = -e \int v\xi\, f_1\, \mathrm{d}^3v = \frac{4\pi}{3} \frac{e^2 E}{T_e} \int_0^\infty \frac{v^4 f_M}{\nu_{ei}(v)}\, \mathrm{d}v$$

With $\tau = 1/\nu_{ei}$, $\langle v^2 \rangle = 3T_e/m_e$:

Key result: Drude with $\tau(v)$

$$\sigma = \frac{n_e e^2}{m_e} \frac{\langle v^2 \tau \rangle}{\langle v^2 \rangle}$$

constant τ: Drude, ch. 2. Coulomb: $\tau \propto v^3$, integrand $v^7 f_M$, peak at $\sqrt{7}\, v_{te} \approx 2.6\, v_{te}$.



The current is carried by the tail.

Conductivity of a Lorentz gas (3): the factor $32/3\pi$

Gaussian moment, $\int_0^\infty v^{2k+1} e^{-v^2/2v_{te}^2} \mathrm{d}v = k!\,(2v_{te}^2)^{k+1}/2$; for $k = 3$:

$$\int_0^\infty v^7 f_M \,\mathrm{d}v = \frac{n_e}{(2\pi)^{3/2} v_{te}^3} \cdot 48\, v_{te}^8$$

With $\nu_{ei} = n_i \Gamma_{ei}/v^3$, $n_i\Gamma_{ei} = 4\pi n_e Z e^4 \ln\Lambda/m_e^2$, $v_{te}^2 = T_e/m_e$:

$$\sigma_L = \frac{4\pi}{3} \frac{e^2}{T_e\, n_i \Gamma_{ei}} \frac{48\, n_e v_{te}^5}{(2\pi)^{3/2}} = \frac{16}{(2\pi)^{3/2}} \frac{T_e^{3/2}}{Z e^2 m_e^{1/2} \ln\Lambda}$$

Simple estimate (ch. 2), $\sigma_0 = n_e e^2/m_e \nu_{ei} = 3T_e^{3/2}/(4\sqrt{2\pi}\, Z e^2 m_e^{1/2} \ln\Lambda)$:

Key result: Lorentz gas

$$\frac{\sigma_L}{\sigma_0} = \frac{16}{(2\pi)^{3/2}} \cdot \frac{4\sqrt{2\pi}}{3} = \frac{32}{3\pi} \approx 3.40$$

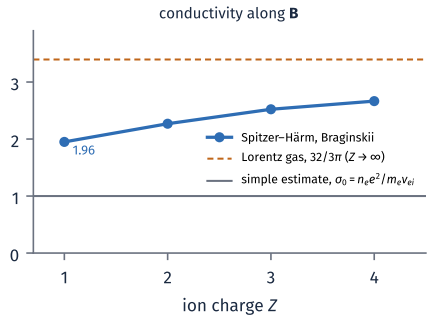
(the fast electrons barely collide)

Electron–electron collisions: Spitzer and Härm

For $Z = 1$, e–e collisions are as frequent as e–i:

- $\boldsymbol{R}_{ee} = 0$: alone they give *no* resistance
- but they pass drift from the fast carriers (weakly scattered by ions) to slow electrons (strongly scattered), and keep the isotropic part Maxwellian: σ drops

Solved numerically (Spitzer & Härm 1953; Braginskii 1965): $\sigma_\parallel/\sigma_0 = 1.96, 2.27, 2.67$ for $Z = 1, 2, 4$, toward 3.40 as $Z \to \infty$.



Key result

$Z = 1$: $\eta_\parallel = 0.51\, m_e \nu_{ei}/n_e e^2$ (Spitzer, ch. 2); across a strong $\boldsymbol{B}$, $\eta_\perp \approx 1.96\,\eta_\parallel$.

Runaway electrons

Slowing down versus deflection

Fast electron, $v \gg v_{te}$: field particles at rest. From module 7a, with $\Gamma_{ee} = 4\pi e^4 \ln\Lambda / m_e^2$, $n_i \Gamma_{ei} = Z n_e \Gamma_{ee}$:

	friction $\lvert\hat{\boldsymbol{v}} \cdot \boldsymbol{A}\rvert$	$B_\perp$
electrons ($m_a = m_b$)	$2 n_e \Gamma_{ee} / v^2$	$n_e \Gamma_{ee} / v$
ions ($m_e / m_i \to 0$)	$Z n_e \Gamma_{ee} / v^2$	$Z n_e \Gamma_{ee} / v$

Change of speed: $\Delta\lvert\boldsymbol{v}\rvert = \hat{\boldsymbol{v}} \cdot \Delta\boldsymbol{v} + \lvert\Delta\boldsymbol{v}_\perp\rvert^2 / 2v$, with $\left\langle \lvert\Delta\boldsymbol{v}_\perp\rvert^2 \right\rangle / \Delta t = 2B_\perp$:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \hat{\boldsymbol{v}} \cdot \boldsymbol{A} + \frac{B_\perp}{v} = -\frac{(2+Z) n_e \Gamma_{ee}}{v^2} + \frac{(1+Z) n_e \Gamma_{ee}}{v^2} = -\frac{n_e \Gamma_{ee}}{v^2}$$

Key result

Of the friction $(2+Z)$, only 1 slows the electron: $F_s = 4\pi n_e e^4 \ln\Lambda / m_e v^2$. The rest, $(1+Z)$, is pitch-angle scattering. Ions deflect, they do not slow.

Critical speed and Dreicer generation

Electron moving along $-\boldsymbol{E}$: it gains speed if $eE > F_s$, i.e. above

$$v_c^2 = \frac{4\pi n_e e^3 \ln\Lambda}{m_e E} = \frac{E_D}{E}\, v_{te}^2, \qquad E_D = \frac{4\pi n_e e^3 \ln\Lambda}{T_e} \quad \text{(ch. 2)}$$

ch. 2 counted the whole friction: $v_c^2 = (2+Z)(E_D/E)v_{te}^2$. Truth in between: the field keeps realigning the electrons, so the runaway region is a *cone* around $-\boldsymbol{E}$. **Dreicer generation** ($E \ll E_D$): collisions feed the tail, electrons diffuse into the cone. Connor & Hastie (1975):

$$\frac{\mathrm{d}n_r}{\mathrm{d}t} \approx C\, n_e \nu_{ee} \left(\frac{E_D}{E}\right)^{3(1+Z)/16} \exp\left[-\frac{E_D}{4E} - \sqrt{\frac{(1+Z)E_D}{E}}\right]$$

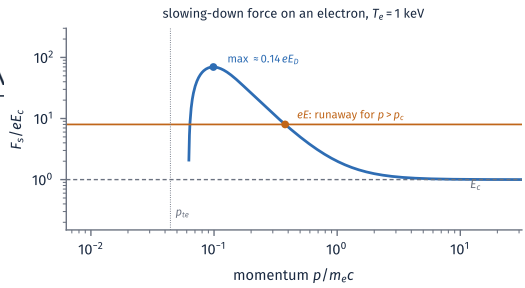
- $E_D/4E$, not the $E_D/2E$ of a Maxwellian at v_c: the field itself raises the tail along its direction
- exponentially sensitive: runaway switches on when E reaches a few per cent of E_D

The relativistic floor: the critical field E_c

$v \to c$ while $p = \gamma m_e v$ grows: the drag stops falling (Connor & Hastie 1975)

$$F_s = eE_c \frac{1+p^2}{p^2} \quad (p \text{ in } m_e c), \qquad E_c = \frac{4\pi n_e e^3 \ln\Lambda}{m_e c^2}$$

- $E < E_c$: no runaway at any energy
- $E > E_c$: runaway above $p_c = m_e c/\sqrt{E/E_c - 1}$
- $n_e = 10^{14}$ cm^{-3}, $\ln\Lambda = 15$: $E_c \approx 0.08$ V/m



Thermal peak $\sim eE_D$; floor eE_c, 511 times lower at 1 keV.

Avalanches: from tokamaks to thunderclouds

Close collisions, neglected by Landau, matter here: a runaway knocks a thermal electron above p_c, which runs away in turn. Exponential growth at a rate $\sim (E/E_c - 1)/(\tau_c \ln \Lambda)$, $\tau_c = m_e c/eE_c$ (Rosenbluth & Putvinski 1997). Needs only $E > E_c$.

Runaway electrons in nature and in the laboratory

- **tokamak disruption**: 10 keV $\longrightarrow$ 10 eV in $\sim$ 1 ms; η up $\times 3 \times 10^4$; E of tens of V/m: $\ll E_D \approx 4$ kV/m but $\gg E_c$. Avalanche: MA beams of tens of MeV electrons (ITER design issue)
- **thunderclouds**: bound electrons of air act as field particles; threshold ≈ 0.3 MV/m at sea level. Avalanches make *terrestrial gamma-ray flashes*, up to tens of MeV (BATSE 1994; RHESSI, AGILE, Fermi)
- **solar flares**: reconnection fields $\gg E_D \approx 5 \times 10^{-3}$ V/m: hard X-rays (ch. 2, ch. 15)

Runaway avalanches above thunderstorms



Simulation of a gamma-ray flash, 0.2, 1.4 and 2.0 ms after onset. NASA/Goddard Space Flight Center/J. Dwyer, Florida Inst. of Technology

- the cloud field drives a relativistic runaway avalanche (yellow): electrons of tens of MeV
- deflected by the nuclei of air, they radiate gamma rays (magenta) by bremsstrahlung, ch. 2: *terrestrial gamma-ray flashes*, sub-millisecond, up to tens of MeV (BATSE 1994; RHESSI, AGILE, Fermi)
- some gamma rays make electron–positron pairs, which spiral up along $\boldsymbol{B}$ to the spacecraft

Check your understanding

1. Which property of U_{ij} gives energy conservation? Which gives momentum conservation?
2. Why can two Maxwellians with different temperatures not be a collisional equilibrium? What about two Maxwellians at the same T drifting relative to each other?
3. In a Lorentz gas, why is any isotropic $f(v)$ an equilibrium? What makes the real electron distribution Maxwellian?
4. Why does a single collision frequency underestimate the conductivity by $32/3\pi$?
5. Electron–electron collisions conserve electron momentum. How can they still change the resistivity?
6. Why do ions deflect a fast electron but hardly slow it down?
7. Is there runaway in a tokamak with $E = 0.05$ V/m and $n_e = 10^{14}$ cm^{-3}?

Exercises in chapter 7 of the notes.

So far, and next

In this module

- Landau operator conserves particles, momentum ($\boldsymbol{R}_{ab} = -\boldsymbol{R}_{ba}$), energy ($u_i U_{ij} = 0$)
- H-theorem: $\dot{H} \leq 0$; equilibrium = Maxwellians with common T, $\boldsymbol{V}$
- Lorentz gas: pitch-angle scattering; $\sigma_L = (32/3\pi)\,\sigma_0$
- e–e collisions: Spitzer, $\sigma_\parallel = 1.96\,\sigma_0$ for $Z = 1$
- runaway: $F_s \propto v^{-2}$, Dreicer rate, floor E_c, avalanches

Next: module 8a. Waves acting back on the particles: quasi-linear theory, trapping, particle-in-cell simulation.