Plasma Physics and Dynamo Theory • Module 7a

Collisions in kinetic theory: the Fokker–Planck equation

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Outline

1. Random kicks: diffusion in velocity space
2. The Fokker–Planck equation in three dimensions
3. Friction and diffusion from Coulomb encounters
4. A test particle in a Maxwellian: Chandrasekhar's function
5. The Landau collision operator

Reading

Helander & Sigmar, ch. 3
Goldston & Rutherford, ch. 11
Rosenbluth, MacDonald & Judd (1957)

Notes: chapter 7, §7.1–7.2

Where we are

Key result: recap

- ch. 2: Coulomb deflections add as a *random walk*, $\mathrm{d}\left\langle\theta^2\right\rangle/\mathrm{d}t = 8\pi n u\, b_{90}^2 \ln\Lambda$; small angles win by $8\ln\Lambda \approx 100$; $\nu_{ei}(v) \propto v^{-3}$
- ch. 2 promised: the friction $F(u)$ with $\psi(x)$, and the Maxwellian average $4\sqrt{2\pi}/3$
- ch. 5–6: Vlasov is collisionless and reversible

Today: turn the random walk of ch. 2 into a term of the kinetic equation, for any distribution of the field particles. The result is the collision operator that turns Vlasov into the Landau equation.

Small-angle scattering as diffusion

Adding collisions to the Vlasov equation

For each species a (charge q_a, mass m_a), colliding with all species b (including $b = a$):

$$\frac{\partial f_a}{\partial t} + \boldsymbol{v} \cdot \nabla f_a + \frac{q_a}{m_a}\left(\boldsymbol{E} + \frac{\boldsymbol{v} \times \boldsymbol{B}}{c}\right) \cdot \frac{\partial f_a}{\partial \boldsymbol{v}} = \sum_b C_{ab}[f_a, f_b]$$

Fields on the left: *smoothed* (ch. 5); C_{ab} collects the graininess.

- Boltzmann's integral counts particles scattered in and out; Coulomb cross section: infinite
- distant, small-angle encounters dominate by $8 \ln \Lambda \sim 100$ (ch. 2): each changes $\boldsymbol{v}$ a little
- many small random changes = diffusion in velocity space

Key result

Fokker–Planck: expand C_{ab} to second order in the jump $\Delta \boldsymbol{v}$. Error $O(1/\ln \Lambda)$, a few per cent.

Random kicks: a random walk in velocity

Independent kicks δ_k at a rate r, with $\langle \delta_k \rangle = 0$, $\left\langle \delta_k^2 \right\rangle = \delta^2$. After a time t ($N = rt$ kicks):

$$v - v_0 = \sum_{k=1}^{N} \delta_k, \qquad \langle v - v_0 \rangle = 0$$

$$\left\langle (v - v_0)^2 \right\rangle = \sum_{k,l} \langle \delta_k \delta_l \rangle = \sum_k \left\langle \delta_k^2 \right\rangle + \sum_{k \neq l} \langle \delta_k \rangle \langle \delta_l \rangle = N\delta^2 = r\delta^2 \, t$$

(independent kicks: the cross terms vanish).

Key result

The spread grows as $\sqrt{t}$, not as t: a random walk, like ink in water. So did the deflections of ch. 2, $\left\langle \theta^2 \right\rangle \propto t$.

Random kicks make diffusion: Fokker–Planck in 1D

The distribution of many such particles obeys the diffusion (heat) equation, with $B = r\delta^2 = \langle \Delta v^2 \rangle / \Delta t$:

$$\frac{\partial f}{\partial t} = \frac{B}{2}\frac{\partial^2 f}{\partial v^2}, \qquad f = \frac{1}{\sqrt{2\pi B t}}\, e^{-(v-v_0)^2/2Bt} \quad \text{(variance } Bt\text{, as above)}$$

Check: $\partial_t f = f\,[-\frac{1}{2t} + \frac{(v-v_0)^2}{2Bt^2}]$ and $\frac{B}{2}\partial_v^2 f = \frac{B}{2} f\,[-\frac{1}{Bt} + \frac{(v-v_0)^2}{B^2 t^2}]$: equal.

If the kicks also have a mean, $\langle \Delta v \rangle = A\,\Delta t$ (friction), the distribution drifts as well:

Key result: Fokker–Planck equation in 1D

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial v}(Af) + \frac{1}{2}\frac{\partial^2}{\partial v^2}(Bf), \qquad A = \frac{\langle \Delta v \rangle}{\Delta t}, \quad B = \frac{\langle \Delta v^2 \rangle}{\Delta t}$$

The mathematics of diffusion equations comes later (module MB).

The Fokker–Planck equation in three dimensions (1)

$\psi(\boldsymbol{v}, \Delta\boldsymbol{v})$: probability that a particle at $\boldsymbol{v}$ jumps by $\Delta\boldsymbol{v}$ in a time Δt (long compared with one encounter, short compared with the evolution of f), normalised: $\int \psi(\boldsymbol{v}, \Delta\boldsymbol{v})\, \mathrm{d}^3\Delta v = 1$. A particle at $\boldsymbol{v}$ at time $t + \Delta t$ was at $\boldsymbol{v} - \Delta\boldsymbol{v}$ at time t, for some jump $\Delta\boldsymbol{v}$:

$$f(\boldsymbol{v}, t + \Delta t) = \int \psi(\boldsymbol{v} - \Delta\boldsymbol{v}, \Delta\boldsymbol{v})\, f(\boldsymbol{v} - \Delta\boldsymbol{v}, t)\, \mathrm{d}^3\Delta v$$

The jumps are small: expand $g(\boldsymbol{v}) \equiv \psi(\boldsymbol{v}, \Delta\boldsymbol{v})\, f(\boldsymbol{v}, t)$ in its *first* argument, at fixed jump (indices $i, j = 1, 2, 3$; repeated indices summed):

$$g(\boldsymbol{v} - \Delta\boldsymbol{v}) = g(\boldsymbol{v}) - \Delta v_i \frac{\partial g}{\partial v_i} + \frac{1}{2}\Delta v_i \Delta v_j \frac{\partial^2 g}{\partial v_i\, \partial v_j} - \ldots$$

The Fokker–Planck equation in three dimensions (2)

Integrate term by term over $\Delta \boldsymbol{v}$; derivatives in $\boldsymbol{v}$ come out of the integral. $\int g \, \mathrm{d}^3\Delta v = f \int \psi \, \mathrm{d}^3\Delta v = f(\boldsymbol{v}, t)$,

$$\int \Delta v_i \, \frac{\partial g}{\partial v_i} \, \mathrm{d}^3\Delta v = \frac{\partial}{\partial v_i}\left(f \int \Delta v_i \, \psi \, \mathrm{d}^3\Delta v \right) = \frac{\partial}{\partial v_i}\left(f \, \langle \Delta v_i \rangle \right)$$

$$\int \tfrac{1}{2} \Delta v_i \Delta v_j \, \frac{\partial^2 g}{\partial v_i \partial v_j} \, \mathrm{d}^3\Delta v = \frac{1}{2} \, \frac{\partial^2}{\partial v_i \partial v_j}\left(f \, \langle \Delta v_i \Delta v_j \rangle \right)$$

Move $f(\boldsymbol{v}, t)$ to the left and divide by Δt:

Key result: Fokker–Planck collision operator

$$C[f] = -\frac{\partial}{\partial v_i}\left(A_i f\right) + \frac{1}{2} \, \frac{\partial^2}{\partial v_i \partial v_j}\left(B_{ij} f\right), \qquad A_i = \frac{\langle \Delta v_i \rangle}{\Delta t}, \quad B_{ij} = \frac{\langle \Delta v_i \Delta v_j \rangle}{\Delta t}$$

A_i: *dynamical friction* (a vector); B_{ij}: *velocity diffusion* (a symmetric tensor).

Friction and diffusion from Coulomb encounters

One encounter, in the relative velocity

Test particle a at $\boldsymbol{v}$, field particle b at $\boldsymbol{v}'$. Relative velocity $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}'$, reduced mass $m_r = m_a m_b/(m_a + m_b)$ (ch. 2):

$$m_a\,\Delta\boldsymbol{v}_a = m_r\,\Delta\boldsymbol{u} \qquad \Rightarrow \qquad \Delta\boldsymbol{v}_a = \frac{m_r}{m_a}\,\Delta\boldsymbol{u}$$

Energy conservation in the centre of mass: $|\boldsymbol{u}|$ does not change, $\boldsymbol{u}$ only turns by θ. Split $\Delta\boldsymbol{u}$ along and across $\boldsymbol{u}$:

$$|\boldsymbol{u} + \Delta\boldsymbol{u}|^2 = u^2 \;\Rightarrow\; 2u\,\Delta u_{\parallel} + \Delta u_{\parallel}^2 + |\Delta\boldsymbol{u}_{\perp}|^2 = 0 \;\Rightarrow\; \Delta u_{\parallel} \simeq -\frac{|\Delta\boldsymbol{u}_{\perp}|^2}{2u}$$

The parallel change is *second* order in the deflection, and always negative.

One encounter: the small-angle result

Small angles ($b \gg b_{90}$, ch. 2): the deflection angle and its scale are

$$\theta = \frac{2 b_{90}}{b}, \qquad b_{90} = \frac{q_a q_b}{m_r u^2}, \qquad |\Delta \boldsymbol{u}_\perp| = u \sin\theta \simeq u\theta$$

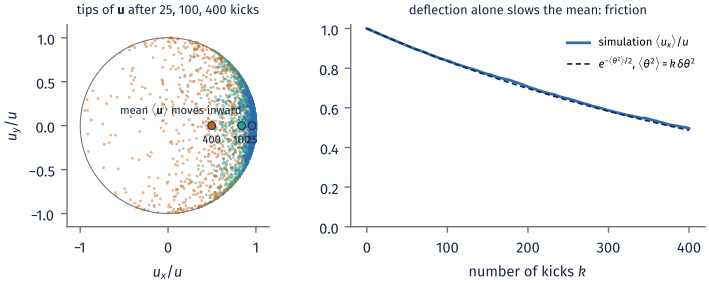
The perpendicular kick points at an azimuth ϕ around $\boldsymbol{u}$, set by where the field particle passes:

Key result

$$\Delta \boldsymbol{u}_\perp = u\theta\,(\cos\phi\, \boldsymbol{e}_1 + \sin\phi\, \boldsymbol{e}_2), \qquad \Delta u_\parallel = -\frac{u\theta^2}{2}$$

$\boldsymbol{e}_1$, $\boldsymbol{e}_2$: unit vectors perpendicular to $\boldsymbol{u}$ and to each other. Check: $u\theta^2/2 = |\Delta \boldsymbol{u}_\perp|^2/2u$, and exactly $\Delta u_\parallel = -u(1-\cos\theta)$, the weight of the momentum-transfer cross section of ch. 2.

Deflection alone produces friction



- each kick only turns $\boldsymbol{u}$: the tip stays on the sphere $|\boldsymbol{u}| = u$
- but the cloud of tips spreads over the sphere, and its *mean* moves inward: $\langle \boldsymbol{u} \rangle$ shrinks
- on average $\langle \Delta u_{\parallel} \rangle = -u \langle \theta^2 \rangle / 2$: friction is the other face of the random walk in angle

Summing over encounters (1): friction

Field particles: a beam of density n at velocity $\boldsymbol{v}'$. In a time Δt, the encounters with impact parameter in $[b, b + \mathrm{d}b]$ and azimuth in $[\phi, \phi + \mathrm{d}\phi]$ number

$$\mathrm{d}N = n\, u\, \Delta t\; b\, \mathrm{d}b\, \mathrm{d}\phi \qquad \text{(particles in a cylinder of length } u\Delta t\text{)}$$

$\Delta u_{\parallel} = -u\theta^2/2$ does not depend on ϕ. Sum over b as in ch. 2, between $b_{\min} \approx b_{90}$ and $b_{\max} \approx \lambda_{\mathrm{D}}$:

$$\int \theta^2\, 2\pi b\, \mathrm{d}b = \int_{b_{\min}}^{b_{\max}} \frac{4b_{90}^2}{b^2}\, 2\pi b\, \mathrm{d}b = 8\pi b_{90}^2 \ln \frac{b_{\max}}{b_{\min}} = 8\pi b_{90}^2 \ln \Lambda$$

$$\frac{\langle \Delta \boldsymbol{u} \rangle}{\Delta t} = -\hat{\boldsymbol{u}}\; nu \int \frac{u\theta^2}{2}\, 2\pi b\, \mathrm{d}b = -4\pi n u^2 b_{90}^2 \ln \Lambda\; \hat{\boldsymbol{u}}$$

The perpendicular kicks average to zero over ϕ: friction acts only along $-\boldsymbol{u}$.

Summing over encounters (2): diffusion

Products of perpendicular kicks: average over ϕ, with $\langle \cos^2\phi \rangle = \left\langle \sin^2\phi \right\rangle = \frac{1}{2}$, $\langle \cos\phi \sin\phi \rangle = 0$:

$$\left\langle \Delta u_{\perp i} \Delta u_{\perp j} \right\rangle_\phi = u^2\theta^2 \left(\langle \cos^2\phi \rangle \, e_{1i} e_{1j} + \left\langle \sin^2\phi \right\rangle e_{2i} e_{2j} \right) = \tfrac{1}{2} u^2 \theta^2 \left(e_{1i} e_{1j} + e_{2i} e_{2j} \right)$$

$\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\hat{\boldsymbol{u}}$ are an orthonormal basis, so $e_{1i} e_{1j} + e_{2i} e_{2j} + \hat{u}_i \hat{u}_j = \delta_{ij}$:

$$\left\langle \Delta u_{\perp i} \Delta u_{\perp j} \right\rangle_\phi = \tfrac{1}{2} u^2 \theta^2 \left(\delta_{ij} - \hat{u}_i \hat{u}_j \right)$$

Sum over b with the same integral $8\pi b_{90}^2 \ln\Lambda$:

$$\frac{\left\langle \Delta u_i \Delta u_j \right\rangle}{\Delta t} = nu \cdot \tfrac{1}{2} u^2 \cdot 8\pi b_{90}^2 \ln\Lambda \left(\delta_{ij} - \hat{u}_i \hat{u}_j \right) = 4\pi n u^3 b_{90}^2 \ln\Lambda \left(\delta_{ij} - \hat{u}_i \hat{u}_j \right)$$

($\Delta u_\parallel^2 \propto \theta^4$ and $\Delta u_\parallel \Delta \boldsymbol{u}_\perp \propto \theta^3$ are dropped: same order as the terms neglected in Fokker–Planck.)

The coefficients for a test particle

Insert $b_{90}^2 = q_a^2 q_b^2 / m_r^2 u^4$ and $\Delta \boldsymbol{v}_a = (m_r/m_a)\Delta \boldsymbol{u}$, and define

$$\Gamma_{ab} = \frac{4\pi q_a^2 q_b^2 \ln\Lambda}{m_a^2}, \qquad U_{ij}(\boldsymbol{u}) = \frac{u^2\delta_{ij} - u_i u_j}{u^3}$$

Key result: one beam of field particles at v', density n

$$A_i = -n\,\Gamma_{ab}\left(1 + \frac{m_a}{m_b}\right)\frac{u_i}{u^3}, \qquad B_{ij} = n\,\Gamma_{ab}\,U_{ij}(\boldsymbol{u}), \qquad \boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}'$$

($m_a/m_r = 1 + m_a/m_b$ in A; $(m_r/m_a)^2$ cancels the m_r^2 of b_{90}^2 in B.)

- diffusion only *across* $\boldsymbol{u}$: $U_{ij}u_j = 0$. It turns the relative velocity, never stretches it
- check with ch. 2: $|A| = \nu v$ with $\nu = n_i\Gamma_{ei}/v^3 = 4\pi n_e Z e^4 \ln\Lambda / m_e^2 v^3$ (electron on ions at rest, $m_e/m_i \to 0$): exactly $\nu_{ei}(v)$ of ch. 2
- heavy test particle ($m_a \gg m_b$): friction enhanced by m_a/m_b, same diffusion

Many field particles

Real field particles have a distribution $f_b(\boldsymbol{v}')$. Each group $f_b(\boldsymbol{v}')\,\mathrm{d}^3v'$ acts as a beam; the contributions add:

$$A_i(\boldsymbol{v}) = -\Gamma_{ab}\left(1+\frac{m_a}{m_b}\right)\int f_b(\boldsymbol{v}')\,\frac{v_i - v_i'}{|\boldsymbol{v}-\boldsymbol{v}'|^3}\,\mathrm{d}^3v', \qquad B_{ij}(\boldsymbol{v}) = \Gamma_{ab}\int f_b(\boldsymbol{v}')\,U_{ij}(\boldsymbol{v}-\boldsymbol{v}')\,\mathrm{d}^3v'$$

- the friction integral has the form of the Coulomb field of a charge density f_b, but in velocity space: next section
- Rosenbluth, MacDonald & Judd (1957) wrote both integrals as derivatives of two "potentials"; this is what numerical Fokker–Planck codes use (notes, §7.2)
- for $b = a$ the coefficients depend on f_a itself: the collision operator is *nonlinear*

A test particle in a Maxwellian

Friction as a Coulomb field in velocity space

Compare with electrostatics:

$$\boldsymbol{A}(\boldsymbol{v}) = -\Gamma_{ab}\left(1+\frac{m_a}{m_b}\right)\int f_b(\boldsymbol{v}')\,\frac{\boldsymbol{v}-\boldsymbol{v}'}{|\boldsymbol{v}-\boldsymbol{v}'|^3}\,\mathrm{d}^3v' \qquad \longleftrightarrow \qquad \boldsymbol{E}(\boldsymbol{r}) = \int \rho(\boldsymbol{r}')\,\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,\mathrm{d}^3r'$$

The friction is (minus) the "electric field" of the field particles: it points *toward* where they are. **Gauss's law**: if f_b is spherically symmetric, the field at radius v is radial and only the "charge" inside counts:

$$\boldsymbol{A} = -\Gamma_{ab}\left(1+\frac{m_a}{m_b}\right)\frac{N_b(<v)}{v^2}\,\hat{\boldsymbol{v}}, \qquad N_b(<v) = \int_0^v 4\pi w^2 f_b(w)\,\mathrm{d}w$$

Key result

Only field particles slower than the test particle cause friction: a uniform spherical shell exerts no force inside it (Newton). The faster ones pull equally in all directions.

Counting the slower particles

Maxwellian field particles: $f_b = n_b \pi^{-3/2} s_b^{-3}\, e^{-v^2/s_b^2}$, $s_b = \sqrt{2T_b/m_b}$; let $x = v/s_b$, $t = w/s_b$:

$$N_b(<v) = \frac{4\pi n_b}{\pi^{3/2} s_b^3} \int_0^v w^2 e^{-w^2/s_b^2}\, \mathrm{d}w = \frac{4n_b}{\sqrt{\pi}} \int_0^x t^2 e^{-t^2}\, \mathrm{d}t$$

By parts, with $t^2 e^{-t^2} = t \cdot (t e^{-t^2})$ and $\int t e^{-t^2} \mathrm{d}t = -\frac{1}{2} e^{-t^2}$:

$$\int_0^x t^2 e^{-t^2}\, \mathrm{d}t = \Big[-\tfrac{1}{2} t e^{-t^2} \Big]_0^x + \frac{1}{2} \int_0^x e^{-t^2}\, \mathrm{d}t = -\frac{1}{2} x e^{-x^2} + \frac{\sqrt{\pi}}{4} \operatorname{erf}(x)$$

Key result: Chandrasekhar function

$$N_b(<v) = n_b\, \psi(x), \qquad \psi(x) = \operatorname{erf}(x) - \frac{2}{\sqrt{\pi}}\, x\, e^{-x^2}$$

$\psi(x)$: fraction of field particles slower than the test particle; $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t$.

The Chandrasekhar function

Small x: expand $\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \dots\right)$ and $x e^{-x^2} = x - x^3 + \dots$:

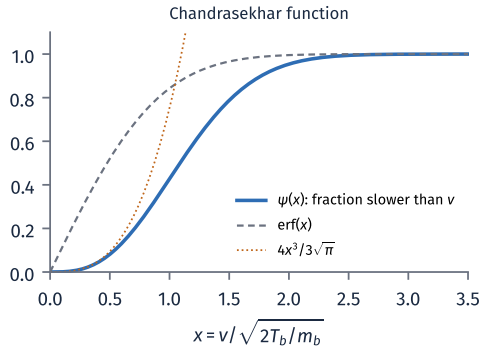
$$\psi \simeq \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} - x + x^3\right) = \frac{4x^3}{3\sqrt{\pi}}$$

the linear terms cancel: few particles are slower.

Large x: $\psi \to 1$, all are slower.

Key result: friction in a Maxwellian

$$\boldsymbol{A} = -\Gamma_{ab}\left(1 + \frac{m_a}{m_b}\right)\frac{n_b\,\psi(x)}{v^2}\,\hat{\boldsymbol{v}}$$



Slow and fast test particles

Slow, $x \ll 1$: $\psi \simeq 4x^3/3\sqrt{\pi}$, friction linear in v, as for a Brownian particle:

$$\boldsymbol{A} \simeq -\nu_s \boldsymbol{v}, \qquad \nu_s = \frac{4}{3\sqrt{\pi}} \Gamma_{ab} \left(1 + \frac{m_a}{m_b}\right) \frac{n_b}{s_b^3}$$

Fast, $x \gg 1$: $\psi \to 1$, the field particles look at rest and the beam result returns:

$$|\boldsymbol{A}| \simeq \Gamma_{ab} \left(1 + \frac{m_a}{m_b}\right) \frac{n_b}{v^2}, \qquad B_\perp = \frac{\Gamma_{ab} n_b}{v} \text{ (from } B_{ij} = n\Gamma_{ab} U_{ij})$$

Key result: friction has a maximum

$\psi(x)/x^2$ peaks at $x \approx 0.97$, i.e. $v \approx 1.4\, v_{tb}$ ($v_{tb} = \sqrt{T_b/m_b}$). Beyond it, the faster a particle, the *less* it is held back: the origin of runaway (ch. 2, module 7b).

Electron–ion friction and ν_{ei}

Electrons drifting at $\boldsymbol{u}$ past ions at rest = an ion moving at $-\boldsymbol{u}$ through Maxwellian electrons. Take a = ion ($Z = 1$), b = electrons, $x = u/\sqrt{2}v_{te}$:

$$m_i|\boldsymbol{A}| = m_i \frac{4\pi e^4 \ln\Lambda}{m_i^2}\left(1 + \frac{m_i}{m_e}\right)\frac{n_e \psi(x)}{u^2} \simeq \frac{4\pi n_e e^4 \ln\Lambda}{m_e u^2}\,\psi(x)$$

By momentum conservation, the electrons feel the same force: this is $F(u)$ of ch. 2. Slow drift, $\psi \simeq 4x^3/3\sqrt{\pi}$:

$$F \simeq \frac{4\pi n_e e^4 \ln\Lambda}{m_e u^2}\,\frac{4}{3\sqrt{\pi}}\,\frac{u^3}{2\sqrt{2}\,v_{te}^3} = \frac{4\sqrt{2\pi}}{3}\,\frac{n_e e^4 \ln\Lambda}{m_e^{1/2} T_e^{3/2}}\,m_e u = m_e \nu_{ei} u$$

Key result

The Maxwellian average replaces the 4π of the estimate by $4\sqrt{2\pi}/3$: the standard ν_{ei} of ch. 2 (Braginskii).

Coulomb friction measured in the laboratory



LEAR at CERN (electron cooling from 1986). CERN PhotoLab, CC BY 4.0

Electron cooling (Budker 1967; Novosibirsk 1974; today ELENA and LEIR at CERN, ESR at GSI)

- a cold electron beam rides with an ion or antiproton beam at the same mean speed
- in the common frame each ion crawls through the electrons: the force we just computed, $\propto \psi(x)/x^2$
- it damps the ions' velocity spread in seconds. The measured force rises linearly, peaks, falls as $1/v^2$

Ion stopping in a hot, dense plasma (OMEGA laser; Frenje et al. 2019)

- protons, tritons, alphas of known energy cross a plasma at $T_e \approx 2$ keV
- the energy loss peaks where $v_{\text{ion}} \approx v_{te}$: the maximum of $\psi(x)/x^2$

The same mathematics in the sky: dynamical friction

Gravity is also inverse-square: $q_a q_b \to -G m_a m_b$. A heavy body M (cluster, black hole) at speed v through stars of mass density ρ, velocity dispersion σ (Chandrasekhar 1943):

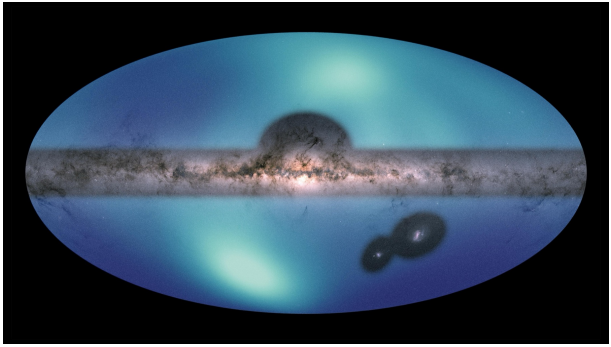
Key result: Chandrasekhar's dynamical friction

$$M\boldsymbol{A} = -\frac{4\pi G^2 M^2 \rho \,\ln\Lambda}{v^2}\,\psi\!\left(\frac{v}{\sqrt{2}\,\sigma}\right)\hat{\boldsymbol{v}} \qquad (M \gg m_b)$$

($M\Gamma_{ab}(1+M/m_b)n_b \to 4\pi G^2 M^2 (n_b m_b)\ln\Lambda$.) The force grows as M^2: massive objects sink first.

- picture: the body focuses stars into a *wake* behind it, which pulls it back
- a $10^6 M_\odot$ cluster at 1 kpc, $v_c = 200$ km/s, sinks to the centre in $t \simeq 1.17\, r^2 v_c / GM \ln\Lambda \approx 5$ Gyr ($\ln\Lambda \approx 10$; Binney & Tremaine). Merging black holes are brought together the same way
- same $\psi(x)$, same $\ln\Lambda$: plasmas and stellar systems share one kinetic theory

Dynamical friction observed: the wake of the LMC

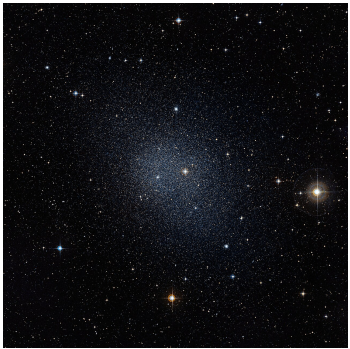


NASA/ESA/JPL-Caltech/Conroy et al. 2021; Milky Way: ESA/Gaia

Density of 1301 halo stars at 60–100 kpc from the centre (Gaia and WISE; Conroy et al. 2021, *Nature*)

- the Large Magellanic Cloud, $\sim 10^{11} M_\odot$ with its dark halo, crosses the Milky Way halo at ~ 300 km/s
- lower half: stars and dark matter focused behind it, the *wake*. Its pull is the friction
- upper half: the Milky Way disc itself pulled off-centre
- the orbit decays: the LMC should merge with the Milky Way in a few Gyr

Dynamical friction at work: clusters, bars, black holes



Fornax dwarf galaxy. ESO/Digitized Sky Survey 2, CC BY 4.0

- **Fornax dwarf** ($\sim$ 140 kpc): its globular clusters, $\sim$ 10 Gyr old, orbit 0.3–1.6 kpc from the centre. In a dark halo with a central cusp, friction would have sunk them in a few Gyr: the *timing problem*. In a cored halo friction stalls: the clusters probe the dark matter
- **the Milky Way bar** has slowed by at least 24% since it formed, measured from the stars it traps in the Hercules stream (Gaia; Chiba & Schönrich 2021): friction against the dark halo
- **massive black holes** of merging galaxies sink to the centre by friction, pair up, and merge by emitting gravitational waves: the nHz background of pulsar timing arrays (NANOGrav 2023)

The Landau collision operator

From Fokker–Planck to Landau (1): flux form

Write C as a divergence. The second-order term, differentiating once:

$$\frac{\partial^2}{\partial v_i \partial v_j}\left(B_{ij} f\right) = \frac{\partial}{\partial v_i}\left[B_{ij}\frac{\partial f}{\partial v_j} + \frac{\partial B_{ij}}{\partial v_j} f\right]$$

so that

Key result

$$C[f] = \frac{\partial J_i}{\partial v_i}, \qquad J_i = \frac{1}{2} B_{ij}\frac{\partial f}{\partial v_j} + \left(\frac{1}{2}\frac{\partial B_{ij}}{\partial v_j} - A_i\right) f$$

J_i is (minus) the flux of particles in velocity space: a diffusive part, $\propto \partial f$, and a drift. The drift is not A_i alone: part of A_i only compensates the variation of B_{ij} with $\boldsymbol{v}$. We now compute it.

From Fokker–Planck to Landau (2): the “fake” friction

Divergence of $U_{ij} = \delta_{ij}/u - u_i u_j/u^3$, using $\partial u/\partial u_j = u_j/u$ and $\partial u_j/\partial u_j = 3$:

$$\frac{\partial}{\partial u_j}\frac{\delta_{ij}}{u} = -\frac{u_i}{u^3}, \qquad \frac{\partial}{\partial u_j}\frac{u_i u_j}{u^3} = \frac{u_i + 3u_i}{u^3} - \frac{3u_i u_j\, u_j}{u^5} = \frac{u_i}{u^3} \qquad \Rightarrow \qquad \frac{\partial U_{ij}}{\partial u_j} = -\frac{2u_i}{u^3}$$

Since $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}'$, $\partial/\partial v_j = \partial/\partial u_j$ at fixed $\boldsymbol{v}'$:

$$\frac{1}{2}\frac{\partial B_{ij}}{\partial v_j} = \frac{\Gamma_{ab}}{2}\int f_b' \frac{\partial U_{ij}}{\partial u_j}\,\mathrm{d}^3 v' = -\Gamma_{ab}\int f_b' \frac{u_i}{u^3}\,\mathrm{d}^3 v' \qquad (f_b' = f_b(\boldsymbol{v}'))$$

and the drift combines:

$$\frac{1}{2}\frac{\partial B_{ij}}{\partial v_j} - A_i = \Gamma_{ab}\left[-1 + \left(1 + \frac{m_a}{m_b}\right)\right]\int f_b' \frac{u_i}{u^3}\,\mathrm{d}^3 v' = \Gamma_{ab}\frac{m_a}{m_b}\int f_b' \frac{u_i}{u^3}\,\mathrm{d}^3 v'$$

The “1” of $1 + m_a/m_b$ is cancelled by the variation of B_{ij}: only m_a/m_b survives.

From Fokker–Planck to Landau (3): one kernel

Write the drift with U_{ij} too. With $\partial/\partial u_j = -\partial/\partial v'_j$ at fixed $\boldsymbol{v}$, then by parts in $\boldsymbol{v}'$ ($f_b \to 0$ at infinity):

$$\int f'_b \, \frac{u_i}{u^3} \, \mathrm{d}^3 v' = -\frac{1}{2} \int f'_b \, \frac{\partial U_{ij}}{\partial u_j} \, \mathrm{d}^3 v' = \frac{1}{2} \int f'_b \, \frac{\partial U_{ij}}{\partial v'_j} \, \mathrm{d}^3 v' = -\frac{1}{2} \int U_{ij} \, \frac{\partial f'_b}{\partial v'_j} \, \mathrm{d}^3 v'$$

Collect: $J_i = \frac{1}{2} \Gamma_{ab} \int U_{ij} \left[f'_b \, \partial_j f_a - (m_a/m_b) \, f_a \, \partial'_j f'_b \right] \mathrm{d}^3 v'$, and $\Gamma_{ab}/2 = 2\pi q_a^2 q_b^2 \ln \Lambda / m_a^2$:

Key result: Landau collision operator (1936)

$$C_{ab}[f_a, f_b] = \frac{2\pi q_a^2 q_b^2 \ln \Lambda}{m_a} \, \frac{\partial}{\partial v_i} \int U_{ij}(\boldsymbol{v} - \boldsymbol{v}') \left[\frac{f_b(\boldsymbol{v}')}{m_a} \, \frac{\partial f_a(\boldsymbol{v})}{\partial v_j} - \frac{f_a(\boldsymbol{v})}{m_b} \, \frac{\partial f_b(\boldsymbol{v}')}{\partial v'_j} \right] \mathrm{d}^3 v'$$

Landau wrote it in 1936, directly from the small-angle limit of Boltzmann's integral; the Fokker–Planck route gives the same operator.

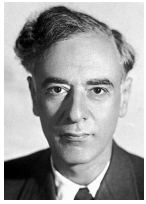
The structure of the Landau operator

The bracket is the operator $\left(\frac{1}{m_a}\frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_b}\frac{\partial}{\partial \boldsymbol{v}'}\right)$ acting on $f_a(\boldsymbol{v}) f_b(\boldsymbol{v}')$, projected by U_{ij} across $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}'$. **Maxwellians at a common temperature are untouched.** With $f_a \propto e^{-m_a v^2/2T}$, $f_b \propto e^{-m_b v'^2/2T}$:

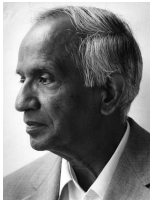
$$\frac{1}{m_a}\frac{\partial f_a}{\partial v_j} f_b' - \frac{1}{m_b} f_a \frac{\partial f_b'}{\partial v_j'} = -\frac{v_j - v_j'}{T} f_a f_b' = -\frac{u_j}{T} f_a f_b', \qquad U_{ij} u_j = 0 \;\Rightarrow\; C_{ab} = 0$$

- the symmetry under $a \leftrightarrow b$, $\boldsymbol{v} \leftrightarrow \boldsymbol{v}'$ gives conservation of particles, momentum and energy, and Boltzmann's H-theorem (module 7b)
- nonlinear when $b = a$: $C_{aa}[f_a, f_a]$. Linearised about a Maxwellian, or with test particles, it is tractable
- accuracy $O(1/\ln\Lambda)$: close collisions are missed, and the cut-off at λ_D is put in by hand. Lenard and Balescu (1960) derived it from BBGKY (ch. 5) with screening

People



L. D. Landau



S. Chandrasekhar[1]



M. N. Rosenbluth[2]

- Landau (1936): the collision integral for Coulomb interactions, ten years before Landau damping
- Chandrasekhar (1943): dynamical friction in star clusters, the function $\psi(x)$
- Rosenbluth, MacDonald & Judd (1957): the Fokker–Planck coefficients as potentials, the basis of numerical codes; Rosenbluth went on to found much of modern fusion theory
- Lenard, Balescu (1960): the operator from first principles, with screening

[1] AIP Emilio Segrè Visual Archives, gift of K. Wali; [2] J. Candy, public domain; Landau: Nobel Foundation, public domain

Check your understanding

1. A particle receives random kicks of zero mean. How does its velocity spread grow with time? Why?
2. Why does a sequence of pure deflections, each conserving $|\boldsymbol{u}|$, produce friction?
3. Show that $U_{ij}u_j = 0$ and $U_{ii} = 2/u$. What do they mean physically?
4. Why is the friction on an electron by ions independent of the ion mass, while that on an ion by electrons carries a factor m_i/m_e?
5. Using Gauss's law, explain why field particles *faster* than the test particle exert no friction on it.
6. Why must the massive black holes of two merging galaxies first be brought together by dynamical friction?

Exercises in chapter 7 of the notes.

So far, and next

In this module

- random kicks = diffusion in velocity space: Fokker–Planck, error $O(1/\ln\Lambda)$
- one encounter: $\Delta u_\parallel = -u\theta^2/2$; deflection makes friction
- beam: $A \propto (1 + m_a/m_b)\,\boldsymbol{u}/u^3$, $B \propto U_{ij}$
- Maxwellian: friction $\propto \psi(x)/v^2$, peak at $v \approx 1.4 v_{tb}$; dynamical friction
- Landau operator, zero for Maxwellians at common T

Next: module 7b. Conservation laws and the H-theorem; the Lorentz gas and Spitzer's conductivity; runaway electrons.