Plasma Physics and Dynamo Theory • Module MB (mathematics)

Partial differential equations and diffusion

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Outline

1. Classification: hyperbolic, parabolic, elliptic
2. The diffusion equation
3. Advection and diffusion
4. Diffusion in velocity: a preview of Fokker–Planck

Reading

Strauss, *Partial Differential Equations*, ch. 1–2
Evans, *PDE*, ch. 2
Risken, *The Fokker–Planck Equation*, ch. 1–3

Notes: Appendix C
Notebook:
`codes/06_diffusion.ipynb`

Why this module

The rest of the course is written in partial differential equations:

- Poisson, $\nabla^2\phi = -4\pi\rho$: Debye shielding (ch. 1), equilibria (ch. 13)
- waves, $\partial_t^2\psi = c^2\nabla^2\psi$: electromagnetic and MHD waves (ch. 11–12)
- Fokker–Planck, $\partial_t f = -\partial_v(Af) + \frac{1}{2}\partial_v^2(Bf)$: collisions (ch. 7, next)
- heat conduction, $\partial_t T = \nabla\cdot(\chi\nabla T)$: transport (ch. 10)
- induction, $\partial_t \boldsymbol{B} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) + \eta\nabla^2\boldsymbol{B}$: MHD, reconnection, dynamos (ch. 12–21)

Questions

- which data determine the solution: initial values, boundary values, both?
- how fast does information travel?
- can the evolution be run backwards in time?

The answers depend only on the *type* of the equation.

Classification

The Cauchy problem for a first-order equation

Recall MA: for Burgers, u is constant along $\mathrm{d}x/\mathrm{d}t = u$. Where does such a curve come from? Take

$$u_t + a\,u_x = 0 \qquad (a \text{ constant: transport at speed } a;\quad a = u:\ \text{Burgers})$$

Cauchy problem: give u on a curve Γ of the (x,t) plane, $x = x(s)$, $t = t(s)$. Can we find u near Γ? On Γ we know $u(s)$, hence its change along the curve,

$$\mathrm{d}u = u_t\,\mathrm{d}t + u_x\,\mathrm{d}x$$

but not u_t and u_x separately: they involve values *off* the curve. With the PDE, two linear equations:

Key result

$$\begin{pmatrix} 1 & a \\ \mathrm{d}t & \mathrm{d}x \end{pmatrix} \begin{pmatrix} u_t \\ u_x \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{d}u \end{pmatrix}, \qquad \Delta = \det = \mathrm{d}x - a\,\mathrm{d}t$$

Everything depends on whether this system can be inverted.

Inverting the system

From the PDE, $u_t = -a\,u_x$. Substitute in the differential: $\mathrm{d}u = (\mathrm{d}x - a\,\mathrm{d}t)\,u_x$. Hence, wherever $\Delta = \mathrm{d}x - a\,\mathrm{d}t \neq 0$,

$$u_x = \frac{\mathrm{d}u}{\mathrm{d}x - a\,\mathrm{d}t}, \qquad u_t = -a\,\frac{\mathrm{d}u}{\mathrm{d}x - a\,\mathrm{d}t}$$

With the parameter s of the curve ($u_s = \mathrm{d}u/\mathrm{d}s$, etc.):

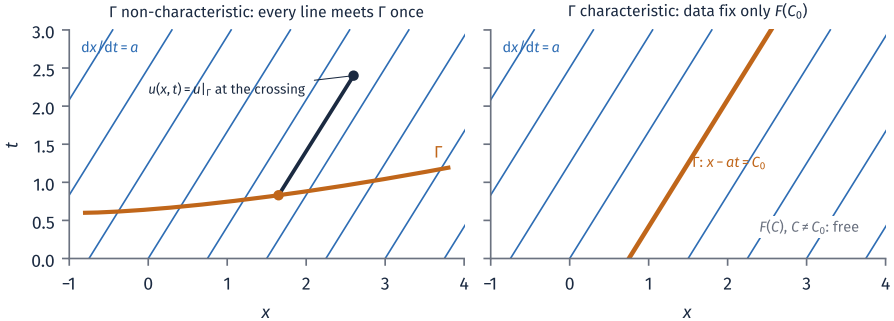
Key result

$$u_x = \frac{u_s}{x_s - a\,t_s}$$

The derivative *across* Γ is reconstructed from the data *on* Γ, unless $x_s - a\,t_s = 0$.

$x_s - a\,t_s = 0$ means $\mathrm{d}x/\mathrm{d}t = a$, Γ tangent to the direction $(a, 1)$: there the data cannot give the transverse derivative. This is the characteristic condition.

The geometry of the Cauchy problem



- **left**: Γ crosses every line $x - at = C$ once. To find $u(x, t)$, follow the line through (x, t) back to Γ and take the value there: the datum is transported along the characteristics
- **right**: Γ is itself a characteristic. Its data say nothing about the other lines

Non-characteristic data: the solution exists and is unique

For constant a, explicitly: if (x_Γ, t_Γ) is where the line through (x, t) meets Γ,

$$u(x, t) = u(x_\Gamma, t_\Gamma), \qquad x - at = x_\Gamma - a\, t_\Gamma$$

Key result: local existence and uniqueness

If Γ is nowhere characteristic and the data are C^1, the method of characteristics gives a unique C^1 solution in a neighbourhood of Γ.

Not "all derivatives on Γ, then Taylor": that works only for analytic data (Cauchy–Kovalevskaya). A C^∞ function is not fixed by its derivatives at a point: e^{-1/x^2} has all of them zero at $x = 0$.

Characteristic data: compatibility and non-uniqueness

If Γ is a characteristic, $\mathrm{d}x/\mathrm{d}t = a$ along it, and by the chain rule

$$\left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{\Gamma} = u_t + \frac{\mathrm{d}x}{\mathrm{d}t}\,u_x = u_t + a\,u_x = 0$$

1. Compatibility: u must be constant on Γ. The data cannot be assigned arbitrarily.

2. Non-uniqueness: even compatible data do not fix the solution off Γ. For constant a the general solution is

Key result

$$u(x,t) = F(x - at), \qquad F \text{ arbitrary;} \qquad \text{characteristics: } x - at = C$$

Data $u = u_0$ on the line $x - at = C_0$ impose only $F(C_0) = u_0$. $F(C)$ for $C \neq C_0$ is free:

$$u = u_0, \qquad u = u_0 + (x - at - C_0), \qquad u = u_0 + \sin(x - at - C_0), \ \ldots$$

all agree on Γ and differ off it; their $u_x = F'(C_0)$ is arbitrary (right panel of the figure above). This is what “the derivatives across Γ stay undetermined” means.

Burgers: the local solution lasts until characteristics cross

$a = u$: characteristics $\mathrm{d}x/\mathrm{d}t = u$, with $\mathrm{d}u/\mathrm{d}t = 0$ along them. Data at $t = 0$ (not characteristic):

$$x(t) = x_0 + u_0(x_0)\, t, \qquad u = u_0(x_0)$$

To know u at (x, t) we must invert $x_0 \mapsto x$, which is possible as long as

$$\frac{\partial x}{\partial x_0} = 1 + u_0'(x_0)\, t > 0. \qquad \text{If } u_0' < 0 \text{ somewhere, this fails at } t_b = \frac{1}{\max_{x_0}\left(-u_0'(x_0)\right)} \quad \text{(MA)}$$

- for $t < t_b$: unique smooth solution, as for linear transport
- at t_b: two characteristics reach the same point carrying different u; $u_x \to \infty$
- for $t > t_b$: only weak solutions; a shock forms

The local theorem is local in time too: it holds while the solution stays smooth and characteristics do not cross.

Characteristics: definition, and discontinuities

Key result: characteristic

A curve on which the Cauchy data do not determine the derivatives off the curve. Along it the PDE reduces to an ODE (here $\mathrm{d}u/\mathrm{d}t = 0$ on $\mathrm{d}x/\mathrm{d}t = a$: exactly the characteristics of Burgers in module MA).

Consequence: across a non-characteristic curve the data fix the derivatives on both sides, so a solution cannot jump there. A discontinuity can only sit on a characteristic:

- **linear** (a fixed): a jump in u_0 at x_0 travels along $x = x_0 + at$, unchanged; the weak solution $u = u_0(x - at)$ of module MA
- **nonlinear** (Burgers): after t_b the shock is a *new* curve, not a characteristic. It moves at the Rankine–Hugoniot speed $s = (u_L + u_R)/2$, and the characteristics of both sides run into it ($u_L > s > u_R$, Lax)

Looking ahead: Δ is the *principal symbol* of the operator on the normal to Γ, $(n_x, n_t) = (\mathrm{d}t, -\mathrm{d}x)$: $n_t + a\, n_x = -(\mathrm{d}x - a\, \mathrm{d}t)$. The same recipe works at any order (next: second order).

Second order: the characteristic equation

Linear, two variables (x, y), constant coefficients:

$$A\,u_{xx} + 2B\,u_{xy} + C\,u_{yy} + (\text{lower-order terms}) = 0$$

Cauchy data on Γ: u *and* its normal derivative, hence u_x, u_y on Γ. Unknowns: u_{xx}, u_{xy}, u_{yy}. Along Γ: $\mathrm{d}(u_x) = u_{xx}\,\mathrm{d}x + u_{xy}\,\mathrm{d}y$ and $\mathrm{d}(u_y) = u_{xy}\,\mathrm{d}x + u_{yy}\,\mathrm{d}y$ are known. With the PDE:

$$\begin{pmatrix} A & 2B & C \\ \mathrm{d}x & \mathrm{d}y & 0 \\ 0 & \mathrm{d}x & \mathrm{d}y \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} -\,\text{l.o.t.} \\ \mathrm{d}(u_x) \\ \mathrm{d}(u_y) \end{pmatrix}, \qquad \det = A\,\mathrm{d}y^2 - 2B\,\mathrm{d}x\,\mathrm{d}y + C\,\mathrm{d}x^2$$

Key result: characteristics of a second-order equation

$$A\,\mathrm{d}y^2 - 2B\,\mathrm{d}x\,\mathrm{d}y + C\,\mathrm{d}x^2 = 0 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

Across these curves the second derivatives can jump (kinks of u travel along them).

Three types of second-order equations

Real roots exist only if $B^2 - AC \geq 0$: the discriminant counts the real families of characteristics.

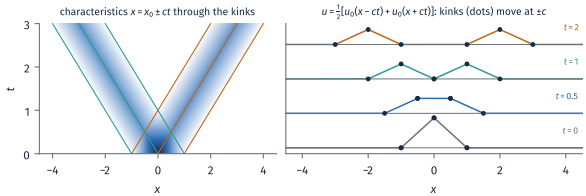
Key result

$B^2 - AC > 0$	**hyperbolic**	two families of characteristics	$u_{tt} = c^2 u_{xx}$
$B^2 - AC = 0$	**parabolic**	one family	$u_t = D\, u_{xx}$
$B^2 - AC < 0$	**elliptic**	none (complex)	$u_{xx} + u_{yy} = 0$

Check, with $y = t$ (only the principal part counts):

- wave, $c^2 u_{xx} - u_{tt} = 0$: $A = c^2$, $B = 0$, $C = -1$: $B^2 - AC = c^2 > 0$
- diffusion, $D\, u_{xx} - u_t = 0$: $A = D$, $B = C = 0$ (u_t is lower order): $B^2 - AC = 0$
- Laplace: $A = C = 1$, $B = 0$: $B^2 - AC = -1 < 0$
- Tricomi, $y\, u_{xx} + u_{yy} = 0$: $B^2 - AC = -y$; hyperbolic for $y < 0$, elliptic for $y > 0$. The type can change from point to point (transonic flow: the sonic line)

Characteristics at work: three examples



- **wave**: $\mathrm{d}x^2 - c^2\mathrm{d}t^2 = 0$, $x = x_0 \pm ct$. The line $t = 0$ is not characteristic: we must give u and u_t there. A tent u_0 has kinks at $x_0 = 0, \pm 1$; they travel along the characteristics (figure)
- **diffusion**: $D\,\mathrm{d}t^2 = 0$, so $t = \text{const}$ (a double root). The initial line *is* a characteristic: u_t cannot be prescribed, the PDE fixes it, $u_t = Du_{xx}$. Hence one datum, and no finite speed
- **Laplace**: no real characteristics, nowhere for a kink to travel: solutions are smooth (analytic) inside the domain, and data are needed on the whole boundary

Hyperbolic: the wave equation, solved

$u_{tt} = c^2 u_{xx}$. Characteristics $\mathrm{d}x/\mathrm{d}t = \pm c$: new variables $\xi = x - ct$, $\eta = x + ct$. Chain rule: $\partial_x = \partial_\xi + \partial_\eta$, $\partial_t = -c\,\partial_\xi + c\,\partial_\eta$, so

$$u_{tt} - c^2 u_{xx} = c^2(\partial_\eta - \partial_\xi)^2 u - c^2(\partial_\eta + \partial_\xi)^2 u = -4c^2\, u_{\xi\eta} = 0$$

$u_{\xi\eta} = 0$ integrates at once: $u = F(\xi) + G(\eta)$,

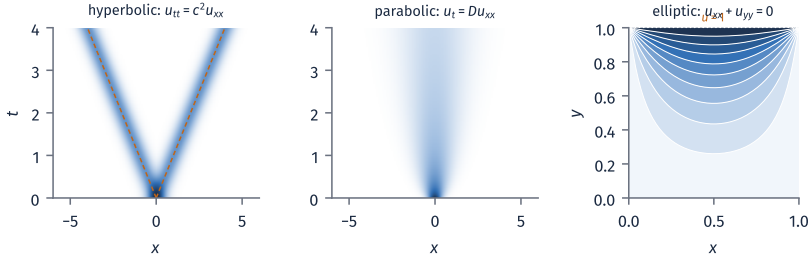
Key result: d'Alembert

$$u(x,t) = F(x - ct) + G(x + ct)$$

two shapes moving rigidly at $\pm c$. With $u(x,0) = u_0$, $u_t(x,0) = 0$: $u = \frac{1}{2}[u_0(x - ct) + u_0(x + ct)]$.

- needs *two* initial data, u and u_t at $t = 0$
- signals travel at finite speed c, along the characteristics
- reversible: $t \to -t$ is again a solution

The three types at a glance



type	data	information travels	backwards in time
hyperbolic	u, u_t at $t = 0$	at speed c, along characteristics	yes
parabolic	u at $t = 0$ (+ boundary)	instantly, but damped	ill-posed
elliptic	u or $\partial_n u$ on the boundary	no time: everywhere at once	–

The diffusion equation

Diffusion from conservation plus Fick's law

Conserved density $u(x,t)$, flux J: the amount in $[a,b]$ changes only through the ends,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u\,\mathrm{d}x = J(a) - J(b) \;\Rightarrow\; \partial_t u + \partial_x J = 0$$

One equation, two unknowns (u, J). A **constitutive law** $J[u]$ closes it: a property of the *material*, not a general principle (like Ohm's $\mathbf{j} = \sigma \boldsymbol{E}$). Simplest: flux down the gradient (Fourier 1822, Fick 1855),

$$J = -D\,\partial u/\partial x, \qquad D > 0 \text{ (diffusion coefficient, cm}^2\,\text{s}^{-1}\text{)}$$

Key result: diffusion equation

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2} \qquad \text{in 3D:} \quad \frac{\partial u}{\partial t} = \nabla\cdot(D\nabla u)$$

Examples: temperature, a tracer, the *magnetic field* in a resistive conductor ($D = \eta$, ch. 2), photons in a star.

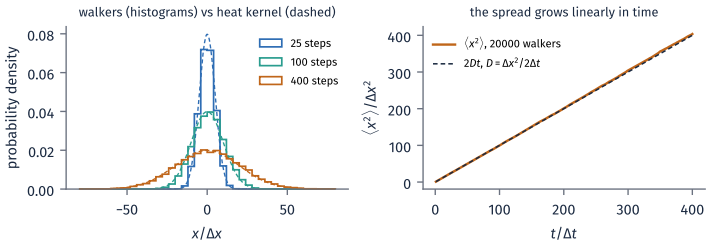
The microscopic origin: a random walk

Each step Δt, a particle jumps $\pm\Delta x$ with equal probability:

$$P(x, t+\Delta t) = \tfrac{1}{2}P(x-\Delta x, t) + \tfrac{1}{2}P(x+\Delta x, t)$$

Taylor-expand both sides, $P(x \pm \Delta x) = P \pm \Delta x\, P_x + \tfrac{1}{2}\Delta x^2 P_{xx} + \ldots$ (odd terms cancel):

$$P + \Delta t\, P_t = P + \tfrac{1}{2}\Delta x^2\, P_{xx} \quad \Rightarrow \quad P_t = D\, P_{xx}, \qquad D = \frac{\Delta x^2}{2\Delta t}$$

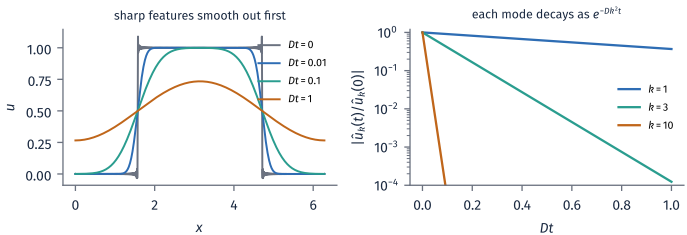


Chapter 2: the direction of a particle performs such a walk under small-angle Coulomb collisions.

Fourier solution: diffusion kills short scales

Write $u = \int \hat{u}(k,t)\, e^{ikx}\, \frac{\mathrm{d}k}{2\pi}$; each mode obeys $\partial_t \hat{u} = -Dk^2 \hat{u}$:

$$\hat{u}(k,t) = \hat{u}(k,0)\, e^{-Dk^2 t}$$



- a feature of size L ($k \sim 1/L$) disappears in the diffusion time $\tau \sim L^2/D$
- backwards in time, $e^{+Dk^2 t}$: the smallest-scale noise explodes. Diffusion is irreversible (unlike Vlasov; collisions make Landau damping irreversible)

What we want: evolve any initial datum

Problem: $u_t = D\,u_{xx}$ on the line, $u(x,0) = u_0(x)$ given. Find $u(x,t)$ for $t > 0$.

1. write the datum as a sum of point sources (module MA): $u_0(x) = \int \delta(x - x')\,u_0(x')\,\mathrm{d}x'$
2. solve once for a unit point at the origin: $G_t = D\,G_{xx}$, $G(x,0) = \delta(x)$
3. the equation does not depend on x: the point at x' evolves into $G(x - x', t)$
4. the equation is linear: the weighted sum of solutions is a solution

Key result: Green's function (propagator)

$$u(x,t) = \int_{-\infty}^{\infty} G(x - x', t)\,u_0(x')\,\mathrm{d}x'$$

G: response to a unit point datum. Once known, any initial-value problem is one integral.

The $\mathcal{L}G = \delta$ of module MA: $(\partial_t - D\partial_x^2)[\theta(t)G] = \delta(t)\,\delta(x)$ (Appendix C).

The heat kernel: computing G

Fourier, $u = \int \hat{u}\, e^{ikx}\, \frac{\mathrm{d}k}{2\pi}$: $\hat{u}(k,t) = \hat{u}(k,0)\, e^{-Dk^2 t}$, and $\delta(x)$ has $\hat{u}(k,0) = 1$:

$$G(x,t) = \int_{-\infty}^{\infty} e^{ikx - Dk^2 t}\, \frac{\mathrm{d}k}{2\pi}$$

1. Complete the square. Expanding $-Dt\big(k - \frac{ix}{2Dt}\big)^2 = -Dtk^2 + ikx + \frac{x^2}{4Dt}$, so

$$ikx - Dk^2 t = -Dt\left(k - \frac{ix}{2Dt}\right)^2 - \frac{x^2}{4Dt}$$

2. Shift the line. Put $q = k - ix/2Dt$: the integral runs along $\operatorname{Im} q = -x/2Dt$. The integrand e^{-Dtq^2} is entire and vanishes as $|\operatorname{Re} q| \to \infty$, so by Cauchy (module MA) the line can be moved back to the real axis. **3. Gaussian integral**, $\int e^{-Dtq^2}\,\mathrm{d}q = \sqrt{\pi/Dt}$:

$$G = \frac{1}{2\pi}\sqrt{\frac{\pi}{Dt}}\; e^{-x^2/4Dt} = \frac{1}{\sqrt{4\pi Dt}}\; e^{-x^2/4Dt}$$

The heat kernel: properties and an example

Key result: heat kernel

$$G(x,t) = \frac{1}{\sqrt{4\pi D t}}\, e^{-x^2/4Dt}, \qquad u(x,t) = \int G(x-x',t)\, u_0(x')\, \mathrm{d}x'$$

- area $\int G\,\mathrm{d}x = \hat{G}(k=0) = 1$ at all t; variance $\langle x^2\rangle = -\partial_k^2 \hat{G}\big|_{k=0} = 2Dt$: the spread grows as $\sqrt{t}$ (random walk!). 3D: $\langle r^2\rangle = 6Dt$
- $G \to \delta(x)$ as $t \to 0^+$: a nascent delta of scaling form, width $\epsilon = \sqrt{2Dt}$ (module MA)
- $G > 0$ everywhere for $t > 0$: infinite speed, but exponentially small beyond $|x| \sim \sqrt{4Dt}$

Example: a step, $u_0 = 1$ for $x < 0$, 0 for $x > 0$. Then, with $s = x - x'$,

$$u(x,t) = \int_{-\infty}^{0} G(x-x',t)\,\mathrm{d}x' = \int_{x}^{\infty} G(s,t)\,\mathrm{d}s = \tfrac{1}{2}\,\mathrm{erfc}\!\left(\frac{x}{\sqrt{4Dt}}\right)$$

The jump becomes at once a front of width $\sim \sqrt{4Dt}$, with $u(0,t) = \frac{1}{2}$.

Diffusion times $\tau \sim L^2/D$

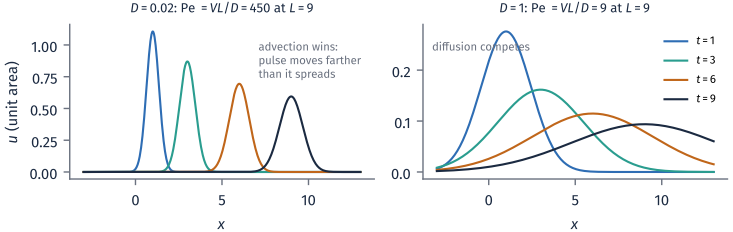
process	D [cm^2 s^{-1}]	L	$\tau = L^2/D$
photons in the Sun ($D = c\ell/3$, $\ell \sim 1$ cm)	10^{10}	$R_\odot = 7 \times 10^{10}$ cm	$\sim 2 \times 10^4$ yr
$\boldsymbol{B}$, base of solar convection zone	5×10^2	2×10^{10} cm	3×10^{10} yr
$\boldsymbol{B}$, Earth's core	10^4	3.5×10^8 cm	$L^2/\pi^2\eta \approx 4 \times 10^4$ yr
1 μm sphere in water (Einstein)	2×10^{-9}	10 μm	$L^2/2D \approx 4$ min

- the light we see left the core thousands of years ago, after $\sim (R/\ell)^2 \sim 10^{22}$ scatterings
- the geomagnetic field is ~ 4 Gyr old, 10^5 decay times: it must be regenerated (dynamo, ch. 16)
- Einstein (1905): $D = k_B T/6\pi\mu a$ for a sphere of radius a in a fluid of viscosity μ; Perrin (1908) measured it and obtained Avogadro's number

Advection and diffusion

Advection plus diffusion: the Péclet number

$\partial_t u + V\,\partial_x u = D\,\partial_x^2 u$. In the frame moving with the flow, $x' = x - Vt$, it is pure diffusion: the pulse drifts at V and spreads as $\sqrt{2Dt}$.



Key result

$$\mathrm{Pe} = \frac{VL}{D} = \frac{\text{diffusion time } L^2/D}{\text{advection time } L/V}$$

$\mathrm{Pe} \gg 1$: carried by the flow; $\mathrm{Pe} \ll 1$: diffused away before it moves.

The induction equation and the magnetic Reynolds number

Faraday + Ohm's law in a moving conductor (ch. 9, 12):

$$\frac{\partial \boldsymbol{B}}{\partial t} = \underbrace{\nabla \times (\boldsymbol{u} \times \boldsymbol{B})}_{\text{advection, stretching}} + \underbrace{\eta \nabla^2 \boldsymbol{B}}_{\text{diffusion}}$$

the same structure, with $D \to \eta = c^2/4\pi\sigma$ (ch. 2). Ratio of the two terms:

Key result

$$R_m = \frac{UL}{\eta} \qquad \text{(magnetic Péclet number)}$$

- $R_m \gg 1$: field frozen into the plasma, carried and stretched (ch. 12); $R_m \ll 1$: it diffuses away in L^2/η
- a dynamo needs R_m above a threshold: stretching must beat diffusion (ch. 16)
- corona $R_m \sim 10^{12}$, ISM 10^{17}, Earth's core 10^3 (ch. 2)

Diffusion in velocity: a preview of Fokker–Planck

Random kicks and friction: the Langevin model

A particle in a gas of lighter particles (Langevin 1908). In a short time Δt many collisions give

$$\Delta v = -\nu v\,\Delta t + \sqrt{2\nu v_t^2\,\Delta t}\;\xi$$

- ν: friction rate [s^{-1}]; v_t: a velocity, identified two slides on with the thermal speed
- ξ: random number, $\langle\xi\rangle = 0$, $\langle\xi^2\rangle = 1$, independent from step to step (*Markov*: the next step depends only on the present v)
- why $\sqrt{\Delta t}$: independent kicks add their variances, so the variance must grow as Δt (random walk)

The Langevin model: drift and diffusion coefficients

Moments of the jump (average over ξ, at fixed v):

$$\langle \Delta v \rangle = -\nu v\, \Delta t, \qquad \langle (\Delta v)^2 \rangle = \nu^2 v^2 \Delta t^2 + 2\nu v_t^2\, \Delta t, \qquad \langle (\Delta v)^n \rangle = O\left(\Delta t^2\right),\ n \geq 3$$

using $\langle \xi \rangle = 0$, $\langle \xi^2 \rangle = 1$: the cross term of the square averages to zero.

Key result: drift and diffusion coefficients

$$A(v) = \lim_{\Delta t \to 0} \frac{\langle \Delta v \rangle}{\Delta t} = -\nu v, \qquad B(v) = \lim_{\Delta t \to 0} \frac{\langle (\Delta v)^2 \rangle}{\Delta t} = 2\nu v_t^2$$

Only these two survive as $\Delta t \to 0$: the friction sets the mean drift, the kicks the spreading. The jumps are small (each is $O\left(\sqrt{\Delta t}\right)$): an expansion in Δv makes sense.

From jumps to the Fokker–Planck equation

$\psi(w, \Delta v)$: probability density of a jump Δv in Δt for a particle at w; $\int \psi \, \mathrm{d}\Delta v = 1$. A particle at v at $t + \Delta t$ was at $w = v - \Delta v$ at t:

$$f(v, t + \Delta t) = \int \psi(v - \Delta v, \Delta v)\, f(v - \Delta v, t)\, \mathrm{d}\Delta v$$

Jumps are small: Taylor-expand the product $g(w) = \psi(w, \Delta v) f(w, t)$ in its first argument, $w = v - \Delta v$,

$$g(v - \Delta v) = g(v) - \Delta v \, \frac{\partial g}{\partial v} + \tfrac{1}{2}(\Delta v)^2 \, \frac{\partial^2 g}{\partial v^2} - \dots$$

(The shift is in the starting point w, at fixed jump Δv: ψ varies smoothly with w.)

From jumps to the Fokker–Planck equation (continued)

Insert in the integral and integrate term by term over Δv; the derivatives ∂_v come out, and

$$\int \psi \, \mathrm{d}\Delta v = 1, \qquad \int \Delta v \, \psi \, \mathrm{d}\Delta v = A \, \Delta t, \qquad \int (\Delta v)^2 \psi \, \mathrm{d}\Delta v = B \, \Delta t$$

give

$$f(v, t + \Delta t) = f - \frac{\partial}{\partial v}(A f)\Delta t + \frac{1}{2}\frac{\partial^2}{\partial v^2}(B f)\Delta t + O\left(\Delta t^2\right)$$

Key result: Fokker–Planck equation ($\Delta t \to 0$)

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial v}\left[A f\right] + \frac{1}{2}\frac{\partial^2}{\partial v^2}\left[B f\right] \overset{\text{Langevin}}{=} \frac{\partial}{\partial v}\left[\nu v f + \nu v_t^2 \frac{\partial f}{\partial v}\right]$$

Advection–diffusion in v: $x \to v$, $V \to A$, $D \to B/2$. The random walk is the case $A = 0$, $\psi = \frac{1}{2}[\delta(\Delta v - \Delta x) + \delta(\Delta v + \Delta x)]$, $B = \Delta x^2/\Delta t = 2D$.

Relaxation to a Maxwellian

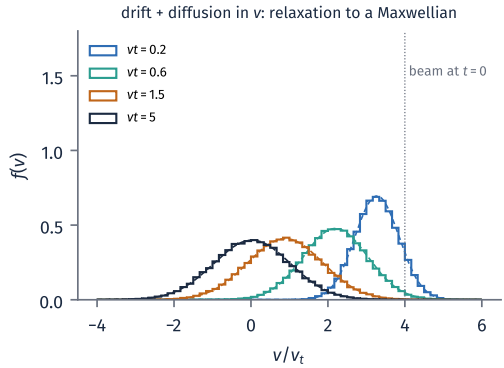
Write the equation as $\partial_t f = -\partial_v S$, with flux in velocity space $S = -\nu v f - \nu v_t^2 \, \partial_v f$.

Equilibrium: $S = 0$,

$$\frac{f'}{f} = -\frac{v}{v_t^2} \implies f \propto e^{-v^2/2v_t^2}$$

the Maxwellian, at the temperature $T = m v_t^2 = mB/2\nu$ set by the balance of kicks and friction (Einstein relation, fluctuation–dissipation).

A beam at v_0 relaxes: mean $v_0 e^{-\nu t}$, variance $v_t^2(1 - e^{-2\nu t})$.



5×10^4 Langevin particles; dashed: exact.

Chapter 7: for Coulomb collisions, A and B come from Rutherford scattering and give the Landau operator.

People


Fourier


Fick


Einstein


Langevin[1]


Fokker


Planck


Kolmogorov[2]

- Fourier (1822): heat equation and Fourier series. Fick (1855): diffusion of solutes
- Einstein (1905), Smoluchowski (1906): Brownian motion as diffusion, $\langle x^2 \rangle = 2Dt$; Perrin (1908): atoms are real
- Langevin (1908): stochastic differential equation. Fokker (1914), Planck (1917): the equation for f
- Kolmogorov (1931): forward and backward equations, the mathematics of diffusion processes

[1] H. Manuel, Wellcome Collection, CC BY 4.0; [2] K. Jacobs, CC BY-SA 2.0 de; others public domain

Check your understanding

1. Classify $u_{tt} + 2u_{tx} + u_{xx} = 0$ and $u_t = u_{xx} + u_{yy}$. Which data do they need?
2. Why does the wave equation need u_t at $t = 0$ while the diffusion equation does not?
3. A temperature profile has structure on scales from 1 m to 1 mm. Which disappears first, and how much faster?
4. Why can't we run the diffusion equation backwards in time? What does this have to do with Landau damping?
5. Estimate R_m for a laboratory liquid-metal experiment: sodium, $\eta \approx 10^3$ cm^2 s^{-1}, $U \sim 10$ m/s, $L \sim 1$ m.
6. In the Langevin model, what fixes the temperature of the final Maxwellian?

Exercises in Appendix C; notebook `codes/06_diffusion.ipynb`.

So far, and next

In this module

- hyperbolic (waves), parabolic (diffusion), elliptic (Poisson): different data, different propagation
- diffusion = conservation + Fick = random walk, $D = \Delta x^2 / 2\Delta t$
- modes decay as $e^{-Dk^2 t}$; heat kernel; $\langle x^2 \rangle = 2Dt$; irreversible
- advection vs diffusion: Pe, R_m
- drift + diffusion in v: Fokker–Planck, relaxation to a Maxwellian

Next: module 7a: collisions in kinetic theory. The Fokker–Planck equation for Coulomb scattering, Rosenbluth potentials, the Landau operator.