Plasma Physics and Dynamo Theory • Module 6b

Resonant particles and kinetic instabilities

Alfio Bonanno
INAF – Osservatorio Astrofisico di Catania

Outline

1. Recap: the plasma dispersion function, exact roots
2. Ion-acoustic waves
3. The physical picture: resonant particles
4. Nonlinear Landau damping: trapping
5. Kinetic instabilities: bump-on-tail, Penrose
6. Type III solar radio bursts

Reading

Chen, ch. 7; Nicholson, ch. 6
Krall & Trivelpiece, ch. 9
O'Neil (1965); Penrose (1960)

Notes: chapter 6, §6.3–6.5

Where we are

Key result: recap

- initial-value problem $\rightarrow$ zeros of the continued $D(k, \omega)$
- weak damping: $D_r(\omega_r) = 0$, $\gamma = -D_i / (\partial D_r / \partial \omega_r)$, with $D_i \propto -f_0'(\omega_r / k)$
- Langmuir waves: $\omega_r^2 = \omega_p^2 + 3k^2 v_t^2$, $\gamma \propto -e^{-1/2k^2\lambda_D^2}$
- today: two species, the physics behind the formula, what happens at large amplitude, and what happens when $f_0' > 0$

$$D(k, \omega) = 1 + \sum_s \chi_s, \qquad \chi_s = \frac{1}{k^2 \lambda_{Ds}^2} \left[1 + \zeta_s Z(\zeta_s) \right], \quad \zeta_s = \frac{\omega}{\sqrt{2} k v_{ts}}$$

Recap: the plasma dispersion function

The plasma dispersion function

For a Maxwellian the integral is a known function. With $\zeta = \omega/\sqrt{2}k v_t$ and $Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_L \frac{e^{-s^2}}{s-\zeta}\, \mathrm{d}s$ (Appendix B):

Key result

$$D(k,\omega) = 1 + \frac{1}{k^2 \lambda_D^2}\left[1 + \zeta\, Z(\zeta)\right]$$

$|\zeta| \gg 1$ (fast waves):
$1 + \zeta Z \simeq -\frac{1}{2\zeta^2} - \frac{3}{4\zeta^4} + i\sqrt{\pi}\zeta e^{-\zeta^2}$

$\Rightarrow$ Bohm–Gross + Landau rate

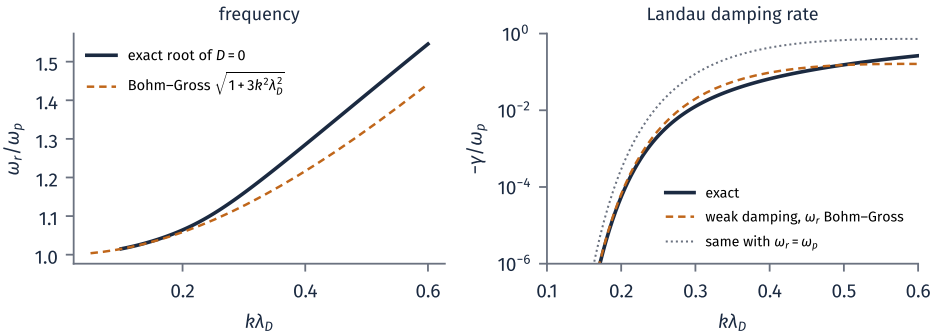
$\zeta \to 0$ (static): $1 + \zeta Z \to 1$,

$D \to 1 + \frac{1}{k^2 \lambda_D^2}$: Debye shielding

($\phi_k = 4\pi q / k^2 D$, ch. 1)

Python: `Z = 1j*sqrt(pi)*wofz(zeta)`; complex zeros of D by Newton's method from Bohm–Gross.

Exact roots



- weak-damping formulas excellent for $k\lambda_D \lesssim 0.3$; at $k\lambda_D = 0.5$ the exact root is $\omega = (1.416 - 0.153i)\,\omega_p$
- the damping rises by 20 orders of magnitude between $k\lambda_D = 0.1$ and 0.5

Ion-acoustic waves

Ion-acoustic waves: the dispersion relation

Slow waves, $k v_{ti} \ll \omega \ll k v_{te}$: electrons are fast ($\zeta_e \to 0$), ions slow ($\zeta_i \gg 1$).

Electrons: $1 + \zeta_e Z(\zeta_e) \to 1$,

$$\chi_e \simeq \frac{1}{k^2 \lambda_{De}^2}$$

Ions: asymptotic Z, as for Langmuir waves,

$$\chi_i \simeq -\frac{\omega_{pi}^2}{\omega^2}\left(1 + \frac{3k^2 v_{ti}^2}{\omega^2}\right)$$

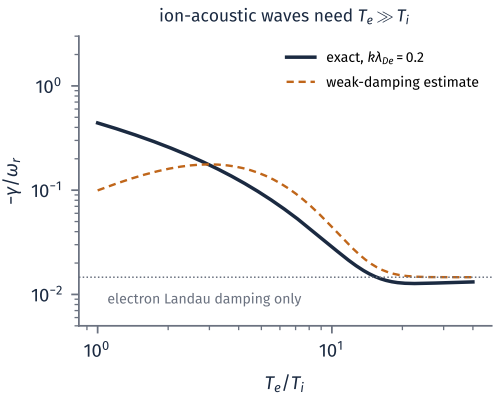
Boltzmann electrons shield (ch. 1).
$D = 0$, with $\omega_{pi}^2 \lambda_{De}^2 = Z T_e / m_i$:

Key result

$$\omega^2 \simeq \frac{k^2 c_s^2}{1 + k^2 \lambda_{De}^2} + 3k^2 v_{ti}^2, \qquad c_s^2 = \frac{Z T_e}{m_i} \qquad \left(\text{long waves: } \frac{\omega}{k} \simeq \sqrt{\frac{Z T_e + 3 T_i}{m_i}}\right)$$

Electron pressure, ion inertia. Short waves ($k\lambda_{De} \gg 1$): $\omega \to \omega_{pi}$.

Ion-acoustic damping: why $T_e \gg T_i$ is needed



Same weak-damping formula, one term per species, at $v_\phi = c_s$:

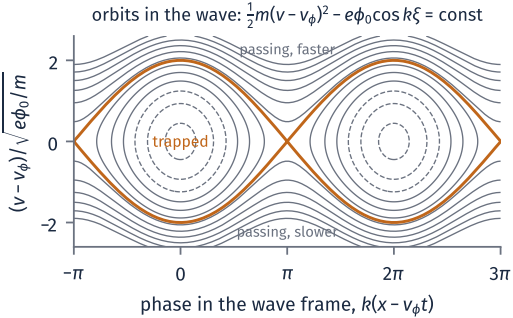
- electrons: $v_\phi \ll v_{te}$, small slope near $v = 0$: $\frac{\gamma_e}{\omega} \simeq -\sqrt{\frac{\pi}{8}}\sqrt{\frac{Z m_e}{m_i}}$
- ions: $v_\phi / v_{ti} = (Z T_e/T_i)^{1/2}$: $\frac{\gamma_i}{\omega} \simeq -\sqrt{\frac{\pi}{8}}\left(\frac{T_e}{T_i}\right)^{3/2} e^{-T_e/2T_i - 3/2}$

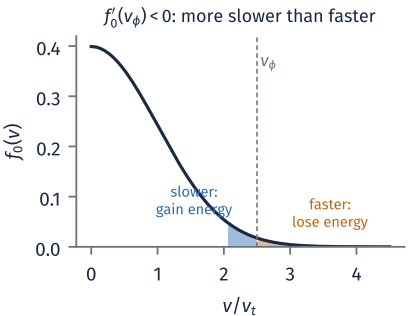
$T_e \sim T_i$: waves damped within a period. Ion-acoustic waves propagate only if $T_e \gtrsim 5$–$10\, T_i$, e.g. after electron heating.

Measured by Wong, Motley & D'Angelo (1964) in a Q-machine; in the solar wind by Helios (Gurnett & Anderson 1977), where $T_e/T_i \sim 1$–3.

The physical picture: resonant particles

Particles in a wave





- in the wave frame the potential is static; particles keep $\frac{1}{2}m(v-v_\phi)^2 - e\phi$
- slightly slower than v_ϕ: pushed forward, gain energy; slightly faster: held back, lose it
- Maxwellian: more particles just below v_ϕ than just above, so the wave loses energy

A rough estimate: reflections

Wave $\phi = \phi_0 \cos(kx - \omega_r t)$, so $E_x = -\partial_x \phi = k\phi_0 \sin(kx - \omega_r t)$: amplitude $E_0 = k\phi_0$. Electrons (charge $-e$), $u = v - v_\phi$; in the wave frame $\frac{1}{2}mu^2 - e\phi_0 \cos k\xi$ is conserved.

Model: a resonant electron is *reflected* in the wave frame, $u \to -u$, i.e. $v \to 2v_\phi - v$:

$$\Delta K_{\text{particle}} = \tfrac{1}{2}m\left[(2v_\phi - v)^2 - v^2\right] = -2mv_\phi(v - v_\phi), \qquad \Delta\mathcal{E}_{\text{wave}} = +2mv_\phi(v - v_\phi)$$

slower electrons ($v < v_\phi$) gain energy, faster ones give it to the wave.

Assumptions (order of magnitude only): reflected band $|u| \lesssim u_m = (2e\phi_0/m)^{1/2}$ (the true separatrix half-width is $2(e\phi_0/m)^{1/2}$); encounters at a rate $|u|/\lambda$. Then

$$\frac{\mathrm{d}W}{\mathrm{d}t} \simeq \frac{2mv_\phi}{\lambda} \int_{v_\phi - u_m}^{v_\phi + u_m} (v - v_\phi)\,|v - v_\phi|\, f_0(v)\,\mathrm{d}v = \frac{2mv_\phi}{\lambda}\left[\int_{v_\phi}^{v_\phi + u_m} - \int_{v_\phi - u_m}^{v_\phi}\right](v - v_\phi)^2 f_0\,\mathrm{d}v$$

A rough estimate: the rate, and what it does not prove

With $f_0(v_\phi + u) \simeq F + Gu$ ($F = f_0(v_\phi)$, $G = f_0'(v_\phi)$): $I_\pm = \frac{F u_m^3}{3} \pm \frac{G u_m^4}{4}$; the F terms cancel:

$$\frac{\mathrm{d}W}{\mathrm{d}t} \simeq \frac{m v_\phi}{\lambda} f_0'(v_\phi)\, u_m^4, \qquad u_m^4 = \frac{4e^2\phi_0^2}{m^2} = \frac{4e^2E_0^2}{m^2k^2}$$

Wave energy (time average, peak E_0): field $E_0^2/16\pi$ + electrons $E_0^2/16\pi$, so $W \simeq E_0^2/8\pi$; $W \propto e^{2\gamma t}$:

$$\gamma = \frac{1}{2W}\frac{\mathrm{d}W}{\mathrm{d}t} \simeq \frac{2}{\pi}\frac{\omega_p^3}{nk^2} f_0'(v_\phi) \qquad \left(\text{kinetic theory: } \frac{\pi}{2}\frac{\omega_p^3}{nk^2} f_0'(v_\phi)\right)$$

Key result

Right sign and scaling, coefficient off by $\pi^2/4 \approx 2.5$: only electrons near v_ϕ exchange energy, with the sign of $f_0'(v_\phi)$. Not a derivation: u_m depends on the amplitude, Landau damping does not.

Reversible damping

Besides the modes, the pole of $\tilde{f}_1 = \frac{i[g+(e/m)\tilde{E}f_0']}{\omega - kv}$ at $\omega = kv$ gives

$$f_1 \supset \left[g(v) + \frac{e}{m}\tilde{E}(kv)\, f_0'(v) \right] e^{ik(x-vt)} \qquad \text{(ballistic term, never decays)}$$

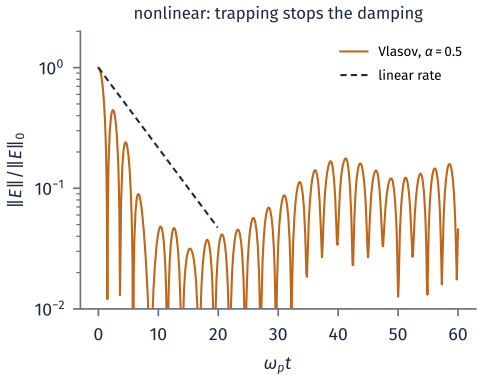
- the *field* decays because the resonant particles' contributions to the charge density phase-mix away, as in module 5b
- the energy lost by the wave is in the fine-scale structure of f near v_ϕ
- it can be recovered: the plasma-wave echo (Malmberg et al. 1968)

Key result

Linear Landau damping holds for times shorter than the trapping time $1/\omega_b$ and the collision time. Collisions smooth the fine structure and make the damping irreversible.

Nonlinear Landau damping: trapping

Large amplitude: trapping

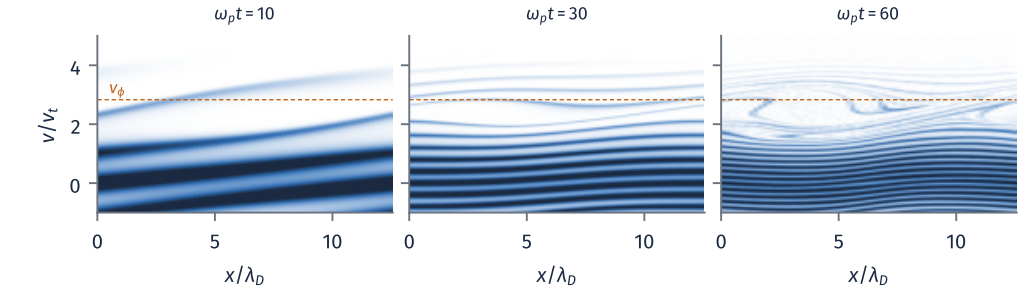


Trapped particles bounce at $\omega_b = (ekE/m)^{1/2}$.

- $\omega_b \ll |\gamma|$: wave damps before anyone bounces: linear theory
- $\omega_b \gtrsim |\gamma|$: trapped particles return the energy after half a bounce; damping stops, amplitude oscillates (O'Neil 1965)

Vlasov run, $f = f_M(1 + 0.5\cos kx)$, $k\lambda_D = 0.5$: initial damping at the linear rate, then the field recovers and saturates.

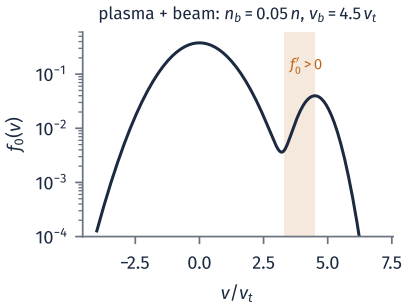
Trapping in phase space

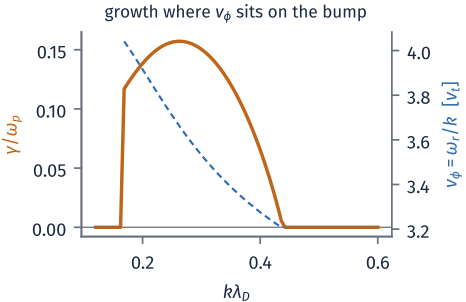


Same run. At $\omega_p t = 10$ the bulk is filamented by phase mixing; electrons near v_ϕ (dashed) are caught in the wave troughs and roll up into vortices, which then persist: a nonlinear BGK-like state. The damping rate of linear theory no longer applies.

Kinetic instabilities: bump-on-tail, Penrose

The bump-on-tail instability





- $f_0'(v_\phi) > 0$: the same formula gives $\gamma > 0$; waves with v_ϕ on the rising side of the bump grow
- warm beam (kinetic): $\gamma \sim \omega_p \frac{n_b}{n}\left(\frac{v_b}{\Delta v}\right)^2$; cold beam (reactive, $\Delta v/v_b \ll (n_b/n)^{1/3}$): $\gamma = \frac{\sqrt{3}}{2^{4/3}}\,\omega_p \left(\frac{n_b}{n}\right)^{1/3}$

When is a distribution unstable? The Penrose criterion

At marginal stability ($\gamma \to 0^+$) both parts of D vanish for real ω:

- $D_i \propto f_0'(v_\phi) = 0$: the phase velocity sits at an extremum v_m of f_0
- $D_r = 0$: $k^2 = \dfrac{\omega_p^2}{n}\,\mathcal{P}\!\int \dfrac{f_0'(v)}{v - v_m}\,\mathrm{d}v = \dfrac{\omega_p^2}{n}\int \dfrac{f_0(v) - f_0(v_m)}{(v - v_m)^2}\,\mathrm{d}v$ (by parts)

A real k needs a positive right-hand side; at a maximum of f_0 the integrand is < 0.

Key result: Penrose (1960)

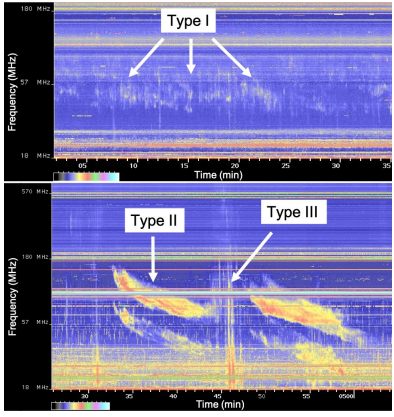
f_0 is unstable if and only if it has a minimum v_m with

$$\int_{-\infty}^{\infty} \frac{f_0(v) - f_0(v_m)}{(v - v_m)^2}\,\mathrm{d}v > 0$$

A single-humped f_0 is always stable; a dip must be deep enough (rigorous proof: Nyquist).

Type III solar radio bursts

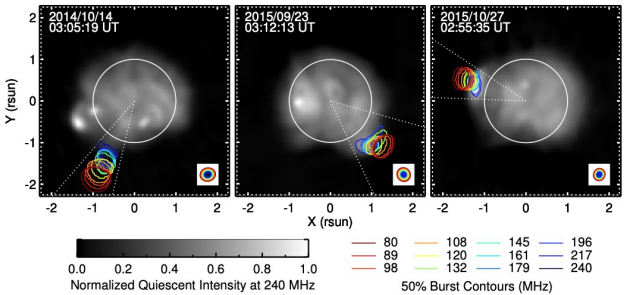
Type III radio bursts



P. McCauley, Learmonth spectrograph, CC BY-SA 4.0

- fast drift from high to low frequency in seconds; classified by Wild & McCready (1950)
- flare electrons ($\sim$ 10 keV, $v \sim c/3$) escape along open field lines: a bump-on-tail in the corona and the wind
- the bump drives Langmuir waves at ω_{pe}; these convert into radio at f_{pe} and $2f_{pe}$ (Ginzburg & Zheleznyakov 1958)
- $f_{pe} = 9\sqrt{n_e}$ kHz falls with height: the drift traces the beam outward (chapter 1)

Seeing the beams



P. McCauley, Murchison Widefield Array, CC BY-SA 4.0

Sources at 240 MHz (blue) to 80 MHz (red) line up outward along the beam path.



Wind (NASA, 1994–). NASA

In situ: beams and Langmuir waves measured together at 1 au (ISEE-3, Lin et al. 1981; Wind; Parker Solar Probe).

Sturrock's dilemma (1964): the bump should relax within ~ 100 km, yet beams reach 1 au. Answer: quasi-linear plateau (chapter 8).

People

- Bohm & Gross (1949): thermal corrections, the first trapping picture
- Malmberg & Wharton (1964): Landau damping measured
- Penrose (1960): stability criterion; O'Neil (1965): nonlinear damping by trapping
- Bernstein, Greene & Kruskal (1957): exact nonlinear waves (BGK modes)
- Mouhot & Villani (2011): Landau damping proved for the full nonlinear Vlasov–Poisson system at small amplitude, for analytic data; Villani's Fields Medal, 2010



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So far, and next

In this module

- ion-acoustic waves, $\omega \simeq k c_s$; strongly ion-damped unless $T_e \gg T_i$
- resonant particles near v_ϕ exchange energy; the sign is that of $f_0'(v_\phi)$
- damping is phase mixing, reversible; trapping stops it when $\omega_b \gtrsim |\gamma|$
- $f_0' > 0$: bump-on-tail; Penrose: a deep enough minimum
- type III bursts: beams, Langmuir waves, radio

Next: module MB: partial differential equations and diffusion, then the Fokker–Planck equation (chapter 7).