

Landau damping

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Outline

1. Landau's initial-value problem
2. Weak damping: Bohm–Gross and the Landau rate
3. The plasma dispersion function and exact roots
4. A numerical experiment, and a real one

Reading

Landau (1946)

Chen, ch. 7; Nicholson, ch. 6

Stix, ch. 8

Notes: chapter 6, §6.1–6.2; Appendix B

Where we are

Key result: recap

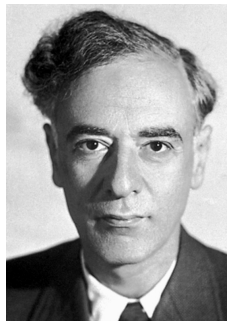
- ▶ module 5b: electrostatic waves from Vlasov,
$$D(k, \omega) = 1 + \frac{4\pi e^2}{mk} \int \frac{f'_0(v)}{\omega - kv} dv = 0$$
- ▶ module MA: $1/(x \mp i0) = \mathcal{P}(1/x) \pm i\pi\delta(x)$; causality picks $\omega + i0$; a Cauchy integral is continued below the axis by the Landau contour
- ▶ free streaming phase-mixes: the density decays while f does not
- ▶ today: put these together for a Maxwellian plasma

Two answers to the same question

Vlasov (1938): look for normal modes $e^{-i\omega t}$, ω real; take the principal value of the singular integral. Result: undamped plasma waves.

Landau (1946): the question is ill-posed. Pose an **initial-value problem**, solve it by Laplace transform, and follow the analytic continuation. Result: the waves are **damped**, without collisions.

The disagreement lasted years: how can a reversible equation produce damping? Settled experimentally by Malmberg & Wharton (1964), conceptually by phase mixing and the echo (module 5b), mathematically by Mouhot & Villani (2011).



L. D. Landau, 1962. Nobel Foundation

The logic of the calculation

1. **Laplace transform** (causality): algebraic equations, $\tilde{E} = N(\omega)/D(k, \omega)$, valid for $\text{Im } \omega > 0$ where every ν -integral is regular.
2. **Analytic continuation** (Landau contour): to push the Bromwich line down, continue N, D to $\text{Im } \omega \leq 0$. Then $E(t) \rightarrow \sum_j R_j e^{-i\omega_j t}$: **the modes are the zeros of the continued D .**
3. **Finding the zeros**: exactly with $Z(\zeta)$, numerically; or, for weak damping (zero just below the axis):
 - evaluate the continued D on the real axis: principal value + $i\pi \times$ residue (Plemelj),
 - Taylor-expand in the small imaginary part γ .

Key result

Plemelj is not a second method: it is the Landau contour with the pole on the real ν axis. Laplace decides the side of the pole; Plemelj computes the integral there.

Landau's initial-value problem

Setting up the initial-value problem

Electrons, ions fixed, 1D: $f = f_0(v) + f_1$, $f_1(x, v, 0) = g(v) e^{ikx}$. Linearised:
 $\partial_t f_1 + ikv f_1 - (e/m)E f'_0(v) = 0$, $ikE = -4\pi e \int f_1 dv$.

Laplace transform, $\tilde{f}(\omega) = \int_0^\infty f e^{i\omega t} dt$, with $\widetilde{\partial_t f_1} = -g - i\omega \tilde{f}_1$:

$$-g - i(\omega - kv)\tilde{f}_1 - \frac{e}{m}\tilde{E}f'_0 = 0 \Rightarrow \tilde{f}_1 = \frac{i}{\omega - kv} \left[g(v) + \frac{e}{m}\tilde{E}f'_0(v) \right]$$

Insert in Poisson and solve for $\tilde{E}$:

Key result

$$\tilde{E}(\omega) = -\frac{4\pi e}{k} \frac{\int \frac{g(v)}{\omega - kv} dv}{D(k, \omega)}, \quad D(k, \omega) = 1 + \frac{4\pi e^2}{mk} \int \frac{f'_0(v)}{\omega - kv} dv$$

Valid for $\text{Im } \omega > 0$: Laplace integral convergent, v -integrals regular.

Inverting: continuation and poles

Bromwich: $E(t) = \frac{1}{2\pi} \int \tilde{E}(\omega) e^{-i\omega t} d\omega$ along $\text{Im } \omega = \sigma > 0$.

To push the contour down, continue numerator and D to $\text{Im } \omega < 0$ with the **Landau contour**:

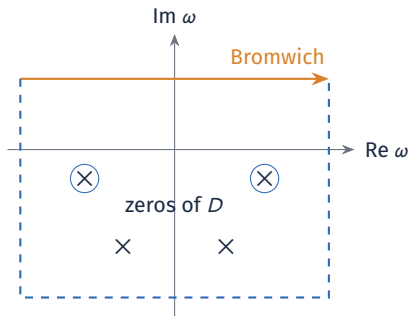
$$\int \frac{\varphi dv}{v-\omega/k} \rightarrow \int \frac{\varphi dv}{v-\omega/k} + 2\pi i \varphi\left(\frac{\omega}{k}\right).$$

For smooth g , f_0 the only singularities are the zeros ω_j of the continued D :

Key result

$$E(t) \xrightarrow{t \rightarrow \infty} \sum_j \frac{N(\omega_j)}{\partial D / \partial \omega|_{\omega_j}} e^{-i\omega_j t}$$

the least damped zero wins.



Weak damping: Bohm–Gross and the Landau rate

Step 3a: the continued D on the real axis

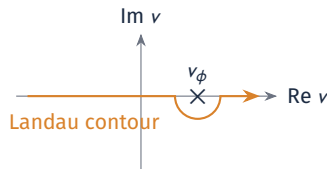
With $\frac{1}{\omega - kv} = -\frac{1}{k} \frac{1}{v - \omega/k}$ and $\omega_p^2 = 4\pi n e^2 / m$:

$$D(k, \omega) = 1 - \frac{\omega_p^2}{nk^2} \int_L \frac{f'_0(v) dv}{v - \omega/k}$$

L : Landau contour, always below the pole $v_\phi = \omega/k$.

Real $\omega = \omega_r$: pole on the axis; L = real axis + half-circle below it, $v - v_\phi = \epsilon e^{i\theta}$, $\theta : \pi \rightarrow 2\pi$:

$$\int_{\text{half circle}} \frac{f'_0 dv}{v - v_\phi} = f'_0(v_\phi) \int_{\pi}^{2\pi} i d\theta = i\pi f'_0(v_\phi)$$



Step 3b: real and imaginary parts of D

The straight parts of L , with the gap $|\nu - \nu_\phi| < \epsilon$ symmetric, give the principal value; the half-circle gives $i\pi f'_0(\nu_\phi)$:

Key result

$$D(k, \omega_r) = D_r + iD_i, \quad D_r = 1 - \frac{\omega_p^2}{nk^2} \mathcal{P} \int \frac{f'_0(\nu) d\nu}{\nu - \nu_\phi}, \quad D_i = -\pi \frac{\omega_p^2}{nk^2} f'_0(\nu_\phi)$$

- ▶ identical to Plemelj, $\frac{1}{\nu - \nu_\phi - i0} = \mathcal{P} \frac{1}{\nu - \nu_\phi} + i\pi\delta(\nu - \nu_\phi)$: the pole sits *above* the contour because $\omega \rightarrow \omega + i0$, the causal prescription of module MA
- ▶ D_r : all the electrons (bulk); D_i : only the **resonant** ones, $\nu = \nu_\phi$
- ▶ the sign of D_i is minus the sign of the slope $f'_0(\nu_\phi)$

Step 3c: the zero just below the axis

The continued D is analytic: Taylor-expand it in the imaginary part. Zero at $\omega = \omega_r + i\gamma$, $|\gamma| \ll \omega_r$:

$$0 = D(\omega_r + i\gamma) \simeq D(\omega_r) + i\gamma \left. \frac{\partial D}{\partial \omega} \right|_{\omega_r}$$

Insert $D = D_r + iD_i$, with $|D_i| \ll |D_r|$; drop $i\gamma \partial D_i / \partial \omega$ (second order):

$$D_r(\omega_r) + i \left[D_i(\omega_r) + \gamma \frac{\partial D_r}{\partial \omega_r} \right] = 0$$

Real and imaginary parts vanish separately:

Key result

$$D_r(\omega_r) = 0 \text{ (gives the frequency),} \quad \gamma = -\frac{D_i(\omega_r)}{\partial D_r / \partial \omega_r} \text{ (gives the damping or growth)}$$

Valid if $|\gamma| \ll \omega_r$ and few resonant particles; otherwise solve $D = 0$ exactly with Z .

The real part: Bohm-Gross

$\mathcal{P}\int$ is dominated by bulk electrons, $v \ll v_\phi = \omega_r/k$. Expand

$\frac{1}{v-v_\phi} = -\frac{1}{v_\phi} \left(1 + \frac{v}{v_\phi} + \frac{v^2}{v_\phi^2} + \frac{v^3}{v_\phi^3} + \dots \right)$ and use, by parts,

$$\int f'_0 dv = 0, \quad \int v f'_0 dv = -n, \quad \int v^2 f'_0 dv = 0, \quad \int v^3 f'_0 dv = -3nv_t^2$$

($v_t^2 = T/m$). Then

$$D_r = 1 - \frac{\omega_p^2}{nk^2} \cdot \frac{1}{v_\phi} \left[\frac{n}{v_\phi} + \frac{3nv_t^2}{v_\phi^3} \right] = 1 - \frac{\omega_p^2}{\omega_r^2} - \frac{3k^2 v_t^2 \omega_p^2}{\omega_r^4}$$

Key result: Bohm-Gross

$$\omega_r^2 \simeq \omega_p^2 + 3k^2 v_t^2 = \omega_p^2 (1 + 3k^2 \lambda_D^2), \quad \frac{\partial D_r}{\partial \omega_r} \simeq \frac{2\omega_p^2}{\omega_r^3} \simeq \frac{2}{\omega_p}$$

The imaginary part: the Landau rate

From $\gamma = -D_i / (\partial D_r / \partial \omega_r)$: $\gamma = \frac{\pi}{2} \frac{\omega_p^3}{n k^2} f'_0 \left(\frac{\omega_r}{k} \right)$. The sign is the slope of f_0 at v_ϕ : damping if $f'_0 < 0$, growth on a bump.

Maxwellian, $f_0 = \frac{n}{\sqrt{2\pi} v_t} e^{-v^2/2v_t^2}$, $f'_0 = -\frac{v}{v_t^2} f_0$, and $\frac{v_\phi^2}{2v_t^2} = \frac{\omega_p^2 + 3k^2 v_t^2}{2k^2 v_t^2} = \frac{1}{2k^2 \lambda_D^2} + \frac{3}{2}$:

Key result: Landau damping of Langmuir waves

$$\gamma \simeq -\sqrt{\frac{\pi}{8}} \frac{\omega_p}{k^3 \lambda_D^3} \exp\left(-\frac{1}{2k^2 \lambda_D^2} - \frac{3}{2}\right)$$

$k\lambda_D = 0.1$: $|\gamma|/\omega_p \sim 10^{-20}$; 0.3 : ~ 0.01 ; 0.5 : ~ 0.1 . Collective waves need $k\lambda_D \ll 1$. The $e^{-3/2}$ is Bohm-Gross in the exponent; with $\omega_r = \omega_p$, $|\gamma|$ is 4.5 times too large.

The plasma dispersion function and exact roots

The plasma dispersion function

For a Maxwellian the integral is a known function. With $\zeta = \omega/\sqrt{2}k v_t$ and $Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_L \frac{e^{-s^2}}{s-\zeta} ds$ (Appendix B):

Key result

$$D(k, \omega) = 1 + \frac{1}{k^2 \lambda_D^2} \left[1 + \zeta Z(\zeta) \right]$$

$|\zeta| \gg 1$ (fast waves):

$$1 + \zeta Z \simeq -\frac{1}{2\zeta^2} - \frac{3}{4\zeta^4} + i\sqrt{\pi}\zeta e^{-\zeta^2}$$

$\Rightarrow$ Bohm-Gross + Landau rate

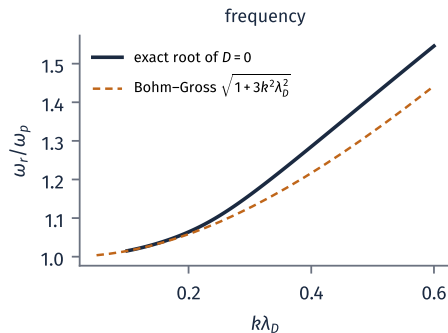
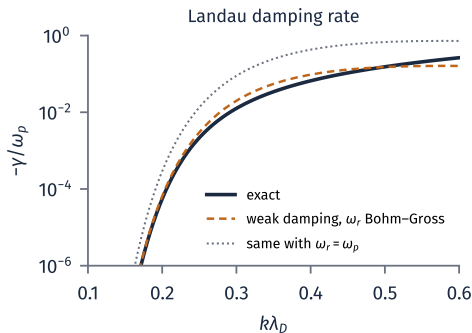
$\zeta \rightarrow 0$ (static): $1 + \zeta Z \rightarrow 1$,

$$D \rightarrow 1 + \frac{1}{k^2 \lambda_D^2} : \text{Debye shielding}$$

$(\phi_k = 4\pi q/k^2 D, \text{ ch. 1})$

Python: `Z = 1j*sqrt(pi)*wofz(zeta)`; complex zeros of D by Newton's method from Bohm-Gross.

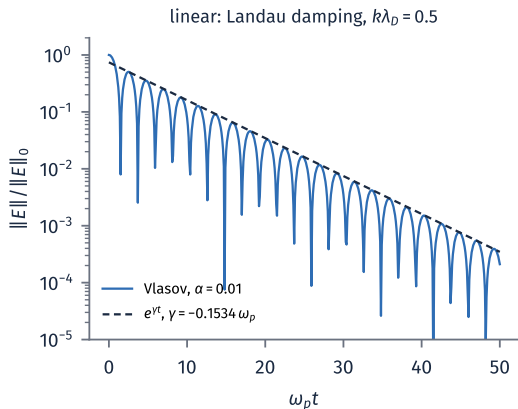
Exact roots



- ▶ weak-damping formulas excellent for $k\lambda_D \lesssim 0.3$; at $k\lambda_D = 0.5$ the exact root is $\omega = (1.416 - 0.153i) \omega_p$
- ▶ the damping rises by 20 orders of magnitude between $k\lambda_D = 0.1$ and 0.5

A numerical experiment, and a real one

A Vlasov simulation



Solve Vlasov–Poisson on a grid in (x, v) : free streaming and acceleration alternate, each an exact shift done with FFTs (Cheng & Knorr 1976). 64×256 points, a laptop second.

Start: $f = f_M(v) (1 + 0.01 \cos kx)$, $k\lambda_D = 0.5$.

Measured: $\gamma = -0.1537 \omega_p$. Root of D : $-0.1534 \omega_p$.

On a velocity grid of spacing Δv the filaments $\cos k(x - vt)$ are resolved until $kt\Delta v \sim 2\pi$: the field comes back at $T_R = 2\pi/k\Delta v$, a numerical echo (here $T_R \approx 270/\omega_p$).

Measuring Landau damping in the laboratory

Malmberg & Wharton (1964, 1966), General Atomic, San Diego:

- ▶ a long, magnetised plasma column, $n \sim 10^8 \text{ cm}^{-3}$, $T_e \sim 10 \text{ eV}$
- ▶ a probe launches an electron plasma wave at fixed frequency; a second probe, moved along the column, measures amplitude and phase
- ▶ spatial damping rate and wavelength versus frequency, compared with the Landau formula using the measured T_e

Agreement over an order of magnitude in damping. Collisions were too rare to explain it.

Historical note

The same group observed the plasma-wave echo in 1968 (module 5b), proving that Landau damping is reversible phase mixing, not dissipation.

In space: Langmuir waves driven by solar electron beams, measured in situ since ISEE-3 (1980s), are the source of type III radio bursts (module 6b).

So far, and next

In this module

- ▶ initial-value problem: $\tilde{E} = N(\omega)/D(k, \omega)$ for $\text{Im } \omega > 0$
- ▶ continue with the Landau contour; zeros of D give the long-time behaviour
- ▶ weak damping: $\omega_r^2 = \omega_p^2 + 3k^2 v_t^2$,
 $\gamma = \frac{\pi}{2} \frac{\omega_p^3}{nk^2} f'_0(v_\phi)$
- ▶ Maxwellian: $\gamma \propto e^{-1/2 k^2 \lambda_D^2}$; exact roots from Z
- ▶ confirmed by simulation and by experiment

Next: module 6b: ion-acoustic waves, the physical picture, nonlinear trapping, and kinetic instabilities.