

Plasma Physics and Dynamo Theory • Module MA (mathematics)

Distributions and singular integrals

Alfio Bonanno

INAF – Osservatorio Astrofisico di Catania

Outline

1. Distributions
2. Weak solutions: Green's functions and shocks
3. Singular integrals: principal value, Sokhotski–Plemelj
4. Laplace transform and analytic continuation

Reading

Lighthill, *Fourier Analysis and Generalised Functions*

Arfken, Weber & Harris, ch. 11, 20

Appel, *Mathematics for Physics and Physicists*, ch. 4, 7

Notes: Appendix B

Notebook:

`codes/05_distributions.ipynb`

Why this module

Module 5b ended with the electrostatic dispersion relation:

$$D(k, \omega) = 1 + \frac{4\pi e^2}{mk} \int_{-\infty}^{\infty} \frac{f'_0(v)}{\omega - kv} dv = 0$$

For real ω the integrand blows up at $v = \omega/k$, where $f'_0 \neq 0$ for a Maxwellian.

Questions

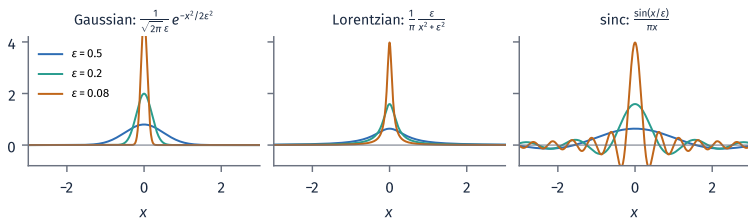
- ▶ what does it mean to integrate through $1/x$?
- ▶ which way around the pole, and why?
- ▶ what are $\delta(v - v_0)$ and $\delta(\omega - kv)$, really?

Tools

- ▶ distributions
- ▶ principal value and Sokhotski–Plemelj
- ▶ Laplace transform, causality, analytic continuation

Distributions

The Dirac δ as a limit



No function has $\delta(x) = 0$ for $x \neq 0$ and $\int \delta = 1$. Instead, take a fixed $\delta_1(y)$ with $\int \delta_1 dy = 1$ and rescale it:

$$\delta_\epsilon(x) = \frac{1}{\epsilon} \delta_1\left(\frac{x}{\epsilon}\right), \quad \int \delta_\epsilon dx = 1 \text{ for every } \epsilon$$

Claim: for every bounded φ continuous at 0, $\lim_{\epsilon \rightarrow 0} \int \delta_\epsilon(x) \varphi(x) dx = \varphi(0)$.

Careful: limits and integrals do not always commute

For the Gaussian and the Lorentzian, $\delta_\epsilon(x) \rightarrow 0$ at every fixed $x \neq 0$. Hence

$$\int \left(\lim_{\epsilon \rightarrow 0} \delta_\epsilon(x) \right) dx = 0 \neq 1 = \lim_{\epsilon \rightarrow 0} \int \delta_\epsilon(x) dx$$

All the area moves into an ever taller, narrower spike, and no integrable function bounds the whole family: the envelope $\sup_\epsilon \delta_\epsilon(x) \sim 1/|x|$ (Lorentzian: $\max_\epsilon \frac{\epsilon/\pi}{x^2+\epsilon^2} = \frac{1}{2\pi|x|}$, at $\epsilon = |x|$) has a divergent integral at $x = 0$.

Key result

Exchanging $\lim$ and $\int$ needs a theorem with checkable hypotheses: **dominated convergence**. The trick is a change of variable that makes a majorant exist.

Passing to the limit: dominated convergence

Step 1: rescale. Put $x = \epsilon y$, $dx = \epsilon dy$:

$$\int \delta_\epsilon(x) \varphi(x) dx = \int \delta_1(y) \varphi(\epsilon y) dy \equiv \int g_\epsilon(y) dy$$

Step 2: pointwise limit. φ is continuous at 0, so for every fixed y : $g_\epsilon(y) \rightarrow \delta_1(y) \varphi(0)$.

Step 3: a majorant independent of ϵ . With $M = \sup |\varphi|$: $|g_\epsilon(y)| \leq M |\delta_1(y)|$, integrable if $\int |\delta_1| dy < \infty$.

Key result: dominated convergence theorem (Lebesgue)

If $g_\epsilon \rightarrow g$ pointwise and $|g_\epsilon| \leq h$ for all ϵ , with $\int h < \infty$, then $\int g_\epsilon \rightarrow \int g$.

Hence $\int \delta_\epsilon \varphi dx \rightarrow \varphi(0) \int \delta_1 dy = \varphi(0)$.

In y the majorant $M|\delta_1(y)|$ exists; in x it did not. That is all Step 1 does.

The three examples, and why test functions are smooth

- ▶ **Gaussian**, $\delta_1 = e^{-y^2/2}/\sqrt{2\pi}$, and **Lorentzian**, $\delta_1 = 1/\pi(1+y^2)$: positive, $\int |\delta_1| = 1$. Dominated convergence applies for any bounded φ continuous at 0.
- ▶ **Sinc**, $\delta_1 = \sin y/\pi y$: $\int |\sin y/y| dy = \infty$. **No majorant**: the theorem fails. Still true for smooth φ , by Fourier: with $\hat{\varphi}(k) = \int \varphi e^{-ikx} dx$ and $\int_{-K}^K e^{ikx} \frac{dk}{2\pi} = \frac{\sin Kx}{\pi x}$, $K = 1/\epsilon$:

$$\int \frac{\sin(x/\epsilon)}{\pi x} \varphi(x) dx = \frac{1}{2\pi} \int_{-1/\epsilon}^{1/\epsilon} \hat{\varphi}(-k) dk \xrightarrow{\epsilon \rightarrow 0} \frac{1}{2\pi} \int \hat{\varphi} dk = \varphi(0)$$

if $\hat{\varphi}$ is integrable, i.e. if φ is smooth enough.

Key result

This is why distribution theory uses smooth, rapidly decaying *test functions*: then every reasonable δ_ϵ converges to δ , oscillating ones included.

Rate of convergence: $\propto \epsilon^2$ (Gaussian), $\propto \epsilon$ (Lorentzian): Appendix B, notebook 05.

Distributions: functions known by their action

Test functions $\varphi(x)$: infinitely differentiable, vanishing fast at infinity.
A **distribution** T is a linear rule $\varphi \mapsto \langle T, \varphi \rangle$, continuous in φ :

$$\langle f, \varphi \rangle = \int f(x) \varphi(x) dx \quad (\text{any integrable } f), \quad \langle \delta, \varphi \rangle = \varphi(0)$$

Derivative, defined so that integration by parts holds (no boundary terms):

Key result

$$\langle T', \varphi \rangle \equiv -\langle T, \varphi' \rangle$$

Heaviside step: $\langle \theta', \varphi \rangle = -\int_0^\infty \varphi' dx = \varphi(0)$, so $\theta' = \delta$; and $\langle \delta', \varphi \rangle = -\varphi'(0)$. Every distribution can be differentiated as often as we like.

Calculus with δ

Change of variable $u = g(x)$ near each simple zero x_i , $du = |g'(x_i)| dx$:

Key result

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}$$

e.g. $\delta(ax) = \delta(x)/|a|$, $x \delta(x) = 0$,
 $\delta(x^2 - a^2) = [\delta(x - a) + \delta(x + a)]/2|a|$.

In 3D, $\delta(\mathbf{x}) = \delta(x)\delta(y)\delta(z)$ and
 $\nabla^2(1/r) = -4\pi \delta(\mathbf{x})$ (Gauss: the flux of
 $-\nabla(1/r)$ through any sphere is 4π).

Already met in this course

- ▶ point charge and Debye shielding (ch. 1)
- ▶ Klimontovich density
 $\sum_i \delta(\mathbf{x} - \mathbf{X}_i) \delta(\mathbf{v} - \mathbf{V}_i)$ (ch. 5)
- ▶ cold beams
 $f_0 = \frac{n}{2} [\delta(\mathbf{v} - \mathbf{v}_0) + \delta(\mathbf{v} + \mathbf{v}_0)]$ (5b)
- ▶ resonance
 $\delta(\omega - k\mathbf{v}) = \delta(\mathbf{v} - \omega/k)/|k|$ (ch. 6, 8)

Weak solutions

Weak solutions

A linear equation $Lu = f$, e.g. $L = \sum_k a_k \frac{d^k}{dx^k}$. Multiply by a test function φ and integrate by parts k times; the boundary terms vanish because φ does:

$$\int (Lu) \varphi \, dx = \int u (L^* \varphi) \, dx, \quad L^* = \sum_k (-1)^k a_k \frac{d^k}{dx^k} \text{ (adjoint)}$$

Key result: weak (distributional) solution

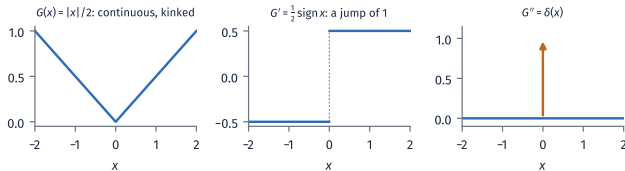
u solves $Lu = f$ weakly if $\int u (L^* \varphi) \, dx = \int f \varphi \, dx$ for all test functions φ

- ▶ all derivatives sit on φ : u needs only be integrable. Kinks, jumps, and $f = \delta$ are allowed
- ▶ classical $\Rightarrow$ weak; smooth weak $\Rightarrow$ classical (Sobolev 1936). Uses: **Green's functions, shocks**

Green's functions are weak solutions

For a linear operator L , a Green's function solves $LG(x, x') = \delta(x - x')$ weakly. Then $u(x) = \int G(x, x')f(x') dx'$ solves $Lu = f$, since $Lu = \int \delta(x - x')f(x') dx' = f(x)$.

Example: $L = d^2/dx^2$ on the line, i.e. $u'' = f$. Then $G(x, x') = G(x - x')$ with $G'' = \delta$:



Weak form ($L^* = L$): is $\int G \varphi'' dx = \varphi(0)$? Split at 0, integrate by parts:

$$\int \frac{|x|}{2} \varphi'' dx = \left[\frac{x}{2} \varphi' \right]_0^\infty - \int_0^\infty \frac{1}{2} \varphi' dx + \left[-\frac{x}{2} \varphi' \right]_{-\infty}^0 + \int_{-\infty}^0 \frac{1}{2} \varphi' dx = \frac{1}{2} \varphi(0) + \frac{1}{2} \varphi(0) = \varphi(0)$$

3D: $-\nabla^2 G = \delta \Rightarrow G = \frac{1}{4\pi r}$ (Coulomb); $(-\nabla^2 + \lambda_D^{-2})G = \delta \Rightarrow G = \frac{e^{-r/\lambda_D}}{4\pi r}$ (Debye, ch. 1).

Conservation laws and characteristics

A conservation law: $\partial_t u + \partial_x F(u) = 0$; the amount of u in $[a, b]$ changes only by the flux F through the ends. **Burgers:** $F = u^2/2$, so $\partial_x F = u \partial_x u$ and

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \quad (\text{simplest model of a steepening sound wave, ch. 14})$$

Method of characteristics. Follow a curve $x(t)$ with $dx/dt = u(x(t), t)$. Along it, by the chain rule,

$$\frac{d}{dt} u(x(t), t) = \frac{\partial u}{\partial t} + \frac{dx}{dt} \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$

So u is constant along the curve, hence so is its slope $dx/dt = u$: the characteristics are **straight lines**, each carrying its initial value,

Key result

$$x = x_0 + u_0(x_0) t, \quad u(x, t) = u_0(x_0) \quad (\text{implicitly: } u = u_0(x - ut))$$

Each point of the initial profile moves at its own speed: high u moves faster than low u .

When do characteristics cross? The breaking time t_b

Two neighbouring characteristics start at x_0 and $x_0 + \Delta x_0$. At time t their separation is

$$\Delta x(t) = [x_0 + \Delta x_0 + u_0(x_0 + \Delta x_0)t] - [x_0 + u_0(x_0)t] \simeq [1 + u'_0(x_0)t] \Delta x_0$$

It vanishes at $t = -1/u'_0(x_0)$: possible only where $u'_0 < 0$ (fast fluid behind slow fluid). The *first* crossing comes from the steepest descent of u_0 :

Key result

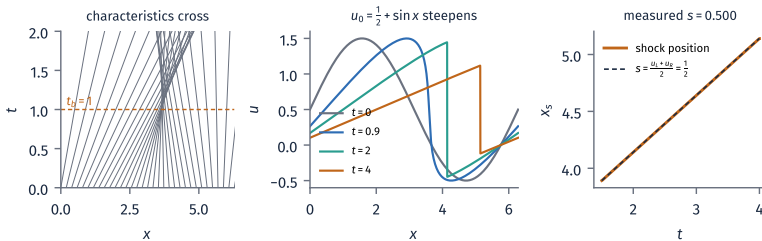
$$t_b = \min_{x_0} \left(-\frac{1}{u'_0(x_0)} \right) = \frac{1}{\max_{x_0} (-u'_0(x_0))}$$

Equivalently, from $u = u_0(x_0)$ and $\partial x / \partial x_0 = 1 + u'_0 t$: $\frac{\partial u}{\partial x} = \frac{u'_0(x_0)}{1 + u'_0(x_0)t} \rightarrow \infty$ as $t \rightarrow t_b$ (gradient catastrophe).

Example: $u_0 = \frac{1}{2} + \sin x$, $u'_0 = \cos x$, steepest descent $-u'_0 = 1$ at $x_0 = \pi$:

$$t_b = 1, \quad x_b = x_0 + u_0(x_0) t_b = \pi + \frac{1}{2} \approx 3.64$$

After t_b : no classical solution, but a weak one



Characteristics cross at $t_b = 1$, $x \approx 3.64$; the profile breaks into a jump moving at constant speed.

Weak form: multiply by a test function $\varphi(x, t)$, integrate over x and $t > 0$, move the derivatives onto φ . In t the boundary term at $t = 0$ brings in the initial data:

$$\iint (u \partial_t \varphi + F(u) \partial_x \varphi) dx dt + \int u_0(x) \varphi(x, 0) dx = 0 \quad \forall \varphi$$

No derivative of u appears: a u with a jump can satisfy it.

The jump condition: Rankine–Hugoniot

A jump at $x_s(t)$, speed $s = \dot{x}_s$, states u_L, u_R . Integral form on $[a, b] \ni x_s$:

$$\frac{d}{dt} \int_a^b u \, dx = F(u(a)) - F(u(b))$$

Shrink $[a, b]$ around the jump: $\frac{d}{dt} [(x_s - a)u_L + (b - x_s)u_R] = s(u_L - u_R) = F_L - F_R$:

Key result: Rankine–Hugoniot

$$s = \frac{F(u_L) - F(u_R)}{u_L - u_R} = \frac{[F]}{[u]}, \quad \text{Burgers: } s = \frac{u_L + u_R}{2}$$

Measured in the simulation: $s = 0.500$, the mean of the two states. Chapter 14: the same condition for mass, momentum and energy gives the jumps across a gas or MHD shock.

Which weak solution? The entropy condition

Weak solutions are **not unique**. For $u_L < u_R$ (here $u_L = 0$, $u_R = 1$) both are weak solutions:

- ▶ a jump moving at $s = \frac{1}{2}$ (it satisfies Rankine–Hugoniot)
- ▶ a smooth fan, $u = x/t$ for $0 < x < t$: a rarefaction wave

The physical one is the limit of vanishing viscosity, $\partial_t u + u \partial_x u = \nu \partial_x^2 u$, $\nu \rightarrow 0$: the fan.

Key result: Lax entropy condition

Characteristics must run *into* the shock: $u_L > s > u_R$. Information is lost in a shock, never created: in gas dynamics, entropy increases across it (ch. 14).

Weak solutions plus a selection rule: the same logic as Landau's contour, where causality picks the right continuation.

Singular integrals

The Cauchy principal value

$\int_{-1}^2 dx/x$ diverges at 0 from both sides, with opposite signs. Cut out a symmetric interval:

Key result

$$\mathcal{P} \int_a^b \frac{\varphi(x)}{x} dx \equiv \lim_{\epsilon \rightarrow 0} \left[\int_a^{-\epsilon} + \int_{\epsilon}^b \right] \frac{\varphi(x)}{x} dx$$

Example:

A regular form, good for computing:

$$\mathcal{P} \int_{-1}^2 \frac{dx}{x} = \lim_{\epsilon \rightarrow 0} [\ln \epsilon - \ln 1 + \ln 2 - \ln \epsilon] = \ln 2$$

$$\mathcal{P} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x} dx = \int_0^{\infty} \frac{\varphi(x) - \varphi(-x)}{x} dx$$

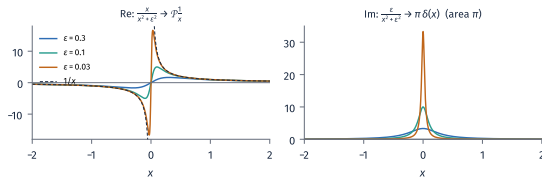
The divergences $\pm \ln \epsilon$ cancel only because the cut is symmetric.

the integrand is finite at 0, $\rightarrow 2\varphi'(0)$.
 $\mathcal{P}(1/x)$ is a distribution.

The Sokhotski–Plemelj formula

Move the singularity off the axis, $1/x \rightarrow 1/(x - i\epsilon)$, and split:

$$\frac{1}{x - i\epsilon} = \frac{x + i\epsilon}{x^2 + \epsilon^2} = \frac{x}{x^2 + \epsilon^2} + i \frac{\epsilon}{x^2 + \epsilon^2}$$



The real part cuts off $1/x$ symmetrically at $|x| \sim \epsilon$; the imaginary part is $\pi \times$ a Lorentzian:

Key result

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{x - x_0 \mp i\epsilon} = \mathcal{P} \frac{1}{x - x_0} \pm i\pi \delta(x - x_0)$$

A worked example, three ways

$$I(a) = \lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{dx}{(x - a - i\epsilon)(1 + x^2)}, \quad a \text{ real.}$$

1. **Principal value**, by partial fractions $\frac{1}{(x-a)(1+x^2)} = \frac{1}{1+a^2} \left[\frac{1}{x-a} - \frac{x+a}{1+x^2} \right]$: the first term has zero principal value, the x term is odd, so

$$\mathcal{P} \int \frac{dx}{(x-a)(1+x^2)} = -\frac{a}{1+a^2} \int \frac{dx}{1+x^2} = -\frac{\pi a}{1+a^2}$$

2. **Plemelj** adds $i\pi \times \frac{1}{1+a^2}$: $I = \frac{\pi(i-a)}{1+a^2} = -\frac{\pi}{a+i}$

3. **Residues**: close the contour above; poles at $x = i$ and $x = a + i\epsilon$:

$$2\pi i \left[\frac{1}{2i(i-a)} + \frac{1}{1+a^2} \right] = \frac{\pi}{i-a} + \frac{2\pi i}{1+a^2} = \frac{\pi(i-a)}{1+a^2} \quad \checkmark$$

Which side of the pole? A driven oscillator

A damped oscillator driven by a force: $\ddot{x} + \gamma \dot{x} + \omega_0^2 x = F(t)/m$, $\gamma > 0$. Fourier components $x, F \propto e^{-i\omega t}$: $\partial_t \rightarrow -i\omega$, so $(-\omega^2 - i\gamma\omega + \omega_0^2) x = F/m$:

$$x(\omega) = \chi(\omega) \frac{F(\omega)}{m}, \quad \chi(\omega) = \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \quad (\text{response function})$$

Poles: $\omega^2 + i\gamma\omega - \omega_0^2 = 0 \Rightarrow \omega_{\pm} = -\frac{i\gamma}{2} \pm \Omega$, $\Omega = \sqrt{\omega_0^2 - \gamma^2/4}$. Both have $\text{Im } \omega_{\pm} = -\gamma/2 < 0$: **below** the real axis.

Response in time: a kick $F = m \delta(t)$ has $F(\omega)/m = 1$, so $x(t)$ is the Green's function

$$G(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi(\omega) e^{-i\omega\tau} d\omega \quad (\text{along the real } \omega \text{ axis})$$

Next: compute it with residues, and see what the position of the poles implies.

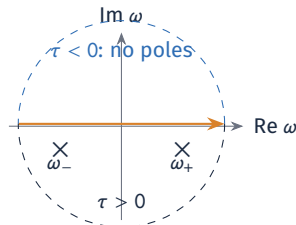
Causality from the position of the poles

$|e^{-i\omega\tau}| = e^{\tau \operatorname{Im} \omega}$: close where it decays.

$\tau < 0$: decays for $\operatorname{Im} \omega > 0$; close **above**: no poles, so $G(\tau < 0) = 0$.

$\tau > 0$: close **below**, clockwise $(-2\pi i)$. With $\chi = -1/[(\omega - \omega_+)(\omega - \omega_-)]$, the residues of $\chi e^{-i\omega\tau}$ are $\mp e^{-i\omega_{\pm}\tau}/2\Omega$:

$$G(\tau) = \frac{-2\pi i}{2\pi} \cdot \frac{-e^{-i\omega_+\tau} + e^{-i\omega_-\tau}}{2\Omega} = e^{-\gamma\tau/2} \frac{\sin \Omega\tau}{\Omega}$$



Key result

Poles below the axis $\Leftrightarrow G(\tau < 0) = 0$: no response before the force (**causality**).

Zero damping: the prescription $\omega \rightarrow \omega + i0$

As $\gamma \rightarrow 0^+$ the poles $\pm\Omega - i\gamma/2$ reach the real axis *from below*. The real-axis contour must stay *above* them, equivalently

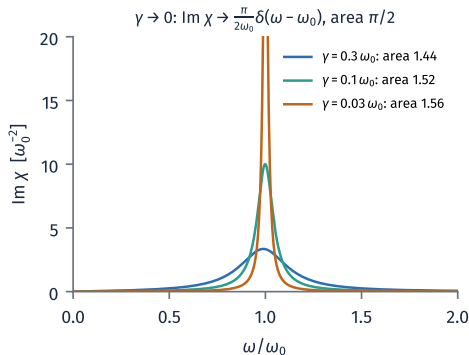
$$\chi = \frac{1}{\omega_0^2 - (\omega + i0)^2}$$

i.e. evaluate the frictionless χ at $\omega + i0$.
Causality chooses the sign.

Near $\omega = \omega_0 > 0$:

$\omega_0^2 - (\omega + i0)^2 \simeq 2\omega_0(\omega_0 - \omega - i0)$, and by Plemelj

$$\chi \simeq \frac{1}{2\omega_0} \left[\mathcal{P} \frac{1}{\omega_0 - \omega} + i\pi \delta(\omega - \omega_0) \right]$$



$\text{Im } \chi$ (absorption) $\rightarrow \frac{\pi}{2\omega_0} \delta(\omega - \omega_0)$:
finite area with zero damping. **Landau damping is the same: resonance plus causality.**

Laplace transform and analytic continuation

The Laplace transform: definition

An initial-value problem: $f(t)$ is given only for $t \geq 0$. Suppose $|f(t)| \leq C e^{\sigma t}$ (it grows at most exponentially). Define

Key result

$$\tilde{f}(\omega) = \int_0^{\infty} f(t) e^{i\omega t} dt$$

Where does it converge? With $\omega = \omega_r + i\omega_i$, $|e^{i\omega t}| = e^{-\omega_i t}$, so $|f e^{i\omega t}| \leq C e^{(\sigma - \omega_i)t}$, integrable on $[0, \infty)$ if $\omega_i > \sigma$: **the transform exists in the half-plane $\text{Im } \omega > \sigma$** , and is analytic there (∂_ω under the integral: the extra factor it does not spoil convergence).

Example (sign convention $e^{-i\omega t}$ for a wave): $f = e^{-i\omega_0 t}$,

$$\tilde{f} = \int_0^{\infty} e^{i(\omega - \omega_0)t} dt = \left[\frac{e^{i(\omega - \omega_0)t}}{i(\omega - \omega_0)} \right]_0^{\infty} = \frac{i}{\omega - \omega_0} \quad (\text{Im } \omega > \text{Im } \omega_0)$$

A mode $e^{-i\omega_0 t}$ becomes a **pole** at $\omega = \omega_0$.

The transform of a derivative: where the initial data come in

Integrate by parts, $\int_0^\infty u \, dv = [uv]_0^\infty - \int_0^\infty v \, du$ with $u = e^{i\omega t}$, $dv = \dot{f} \, dt$:

$$\tilde{f}(\omega) = \int_0^\infty \dot{f} e^{i\omega t} \, dt = \left[f(t) e^{i\omega t} \right]_{t=0}^{t=\infty} - \int_0^\infty f(i\omega) e^{i\omega t} \, dt$$

Upper limit: $|f e^{i\omega t}| \leq C e^{(\sigma - \omega_i)t} \rightarrow 0$. Lower limit: $f(0) e^0 = f(0)$. Hence

Key result

$$\tilde{f} = -f(0) - i\omega \tilde{f}, \quad \tilde{\dot{f}} = -\dot{f}(0) - i\omega \tilde{f} = -\dot{f}(0) + i\omega f(0) - \omega^2 \tilde{f}$$

Meaning: a Fourier transform over all t would give just $-i\omega \hat{f}$. The Laplace integral starts at $t = 0$, and its lower boundary contributes the value of f there: the **initial condition** appears in the transformed equation by itself.

Example: $\dot{x} = -\gamma x$, $x(0) = x_0$. Transform: $-x_0 - i\omega \tilde{x} = -\gamma \tilde{x}$, an *algebraic* equation:

$$\tilde{x} = \frac{x_0}{\gamma - i\omega} = \frac{i x_0}{\omega + i\gamma} \quad (\text{pole at } \omega = -i\gamma) \quad \longleftrightarrow \quad x(t) = x_0 e^{-\gamma t}$$

Inverting: the Bromwich integral

Fix $\sigma' > \sigma$ and let $g(t) = f(t) e^{-\sigma' t}$ for $t \geq 0$, $g = 0$ for $t < 0$: an ordinary, decaying function. On the **horizontal line** $\omega = u + i\sigma'$, with u real from $-\infty$ to ∞ ,

$$\tilde{f}(u + i\sigma') = \int_0^\infty f(t) e^{i(u+i\sigma')t} dt = \int_{-\infty}^\infty g(t) e^{iut} dt = \hat{g}(u),$$

the ordinary Fourier transform of g . Fourier inversion, $g(t) = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{g}(u) e^{-iut} du$, then multiply by $e^{\sigma' t}$ to get back $f = e^{\sigma' t} g$:

Key result: Bromwich integral

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^\infty \tilde{f}(u + i\sigma') e^{-i(u+i\sigma')t} du \equiv \frac{1}{2\pi} \int_{i\sigma' - \infty}^{i\sigma' + \infty} \tilde{f}(\omega) e^{-i\omega t} d\omega$$

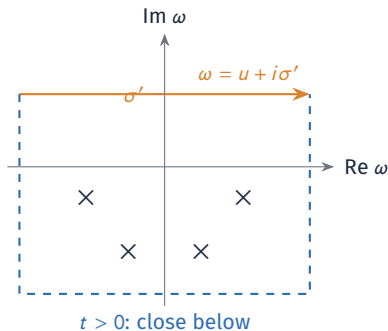
The short notation means: integrate along the line $\text{Im } \omega = \sigma'$, parallel to the *real* axis, from $\text{Re } \omega = -\infty$ to $+\infty$. The $i\sigma'$ is the height of the line, not a direction.

Closing the contour

$$|e^{-i\omega t}| = e^{t \operatorname{Im} \omega}.$$

- ▶ $t < 0$: decays for $\operatorname{Im} \omega \rightarrow +\infty$. Close **above**, where $\tilde{f}$ is analytic (the line is above all singularities): $f(t) = 0$. The transform is causal.
- ▶ $t > 0$: decays for $\operatorname{Im} \omega \rightarrow -\infty$. Close **below**: the large arc contributes nothing (Jordan's lemma), and the integral is $-2\pi i \times$ the residues enclosed (clockwise).

Why above all singularities? It is the only region where the defining integral of $\tilde{f}$ converges; below, $\tilde{f}$ is known only by continuation.



Example: poles give the long-time behaviour

$\ddot{x} + \omega_0^2 x = 0$, $x(0) = x_0$, $\dot{x}(0) = 0$. Transform twice: $\tilde{\ddot{x}} = -\dot{x}(0) + i\omega x(0) - \omega^2 \tilde{x}$, so

$$\tilde{x}(\omega) = \frac{-i\omega x_0}{\omega_0^2 - \omega^2} = \frac{i\omega x_0}{(\omega - \omega_0)(\omega + \omega_0)}$$

For $t > 0$, $e^{-i\omega t}$ decays for $\text{Im } \omega < 0$: close the contour below (clockwise, $-2\pi i$):

$$x(t) = \frac{-2\pi i}{2\pi} \sum \text{Res}[\tilde{x} e^{-i\omega t}] = -i \left[\frac{ix_0}{2} e^{-i\omega_0 t} + \frac{ix_0}{2} e^{i\omega_0 t} \right] = x_0 \cos \omega_0 t \quad \checkmark$$

Key result

Each pole ω_j of $\tilde{f}$ contributes $e^{-i\omega_j t}$. Poles below the real axis decay, poles above grow: **the long-time behaviour is set by the singularities of $\tilde{f}$** , wherever they lie.

To find poles below the axis, $\tilde{f}$ must be known there, not only where the integral converges.

Functions defined by integrals, and their continuation

$F(\omega) = \int \frac{\varphi(v)}{v-\omega} dv$ is analytic above and below the real axis, but jumps across it (Plemelj):

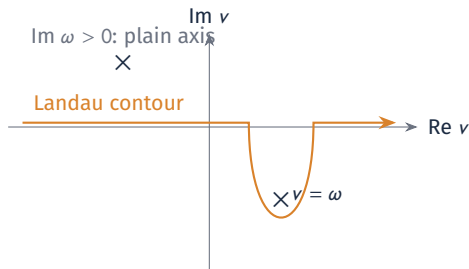
$$F(x+i0) - F(x-i0) = 2\pi i \varphi(x).$$

To continue F from above, keep the pole on the same side: as ω moves below the axis, the contour dips **below** $v = \omega$, i.e. real axis plus a circle around the pole:

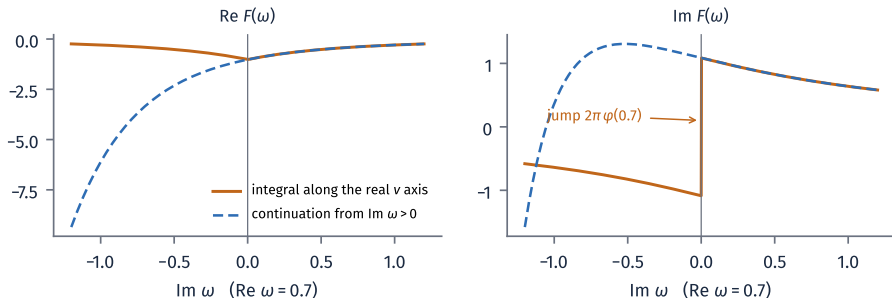
Key result

$$F_{\text{cont}}(\omega) = \int_{-\infty}^{\infty} \frac{\varphi(v) dv}{v - \omega} + 2\pi i \varphi(\omega)$$

for $\text{Im } \omega < 0$ (φ analytic)



Continuation in practice: a Gaussian



$\varphi(v) = e^{-v^2}/\sqrt{\pi}$, F along the vertical line $\text{Re } \omega = 0.7$. The integral along the real axis (orange) jumps by $2\pi i\varphi(0.7)$ when ω crosses the axis; the continuation (dashed) is smooth and is the function whose zeros we want below the axis.

F_{cont} for a Gaussian is the **plasma dispersion function** $Z(\omega)$ of Fried & Conte (1961), computed with the Faddeeva function: in Python, `1j*sqrt(pi)*wofz(w)`.

The recipe for chapter 6

1. causality: $\tilde{f}(\omega)$, $D(k, \omega)$ are first defined for $\text{Im } \omega > 0$, where the ν -integral is regular
2. continue D to $\text{Im } \omega \leq 0$ by deforming the ν -contour below the pole $\nu = \omega/k$ (the Landau contour)
3. the zeros of the continued D are the normal modes; $\text{Im } \omega < 0$ means damping

For weak damping, $\omega = \omega_r + i\gamma$, $|\gamma| \ll \omega_r$, Plemelj does it all:

$$\int \frac{f'_0(\nu) d\nu}{\nu - \omega/k} \simeq \mathcal{P} \int \frac{f'_0(\nu) d\nu}{\nu - \omega_r/k} + i\pi f'_0\left(\frac{\omega_r}{k}\right)$$

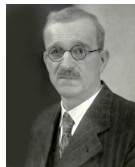
Key result

The real part gives the oscillation frequency (Bohm–Gross); the $i\pi f'_0$ term gives $\gamma \propto f'_0(\omega_r/k)$: damping where f_0 decreases, growth on a bump.

People



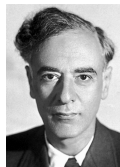
Heaviside



Plemelj



Dirac¹



Landau¹



Sobolev²



Schwartz³

- ▶ Heaviside (1890s): operational calculus, the step function, impulses in circuits
- ▶ Sokhotski (1873), Plemelj (1908): boundary values of Cauchy integrals
- ▶ Dirac (1927, 1930): δ for continuous spectra in quantum mechanics; Bromwich (1916): the inversion contour
- ▶ Sobolev (1936): generalised derivatives and weak solutions of PDEs; Schwartz (1945–50): the theory of distributions, Fields Medal 1950
- ▶ Landau (1946): the initial-value problem for plasma waves, solved with these tools

¹ Nobel Foundation; ² K. Jacobs, CC SA 1.0; ³ K. Jacobs, CC BY-SA 2.0 de; others public domain

So far, and next

In this module

- ▶ distributions act on test functions; δ , $\theta' = \delta$, $\delta(g(x))$
- ▶ weak solutions: Green's functions; shocks, Rankine–Hugoniot, entropy condition
- ▶ $\mathcal{P}\int$: symmetric cut-off;
 $1/(x \mp i0) = \mathcal{P}(1/x) \pm i\pi\delta(x)$
- ▶ causality fixes $\omega \rightarrow \omega + i0$; vanishing damping, finite absorption
- ▶ Laplace transform: long-time behaviour from the singularities
- ▶ continuation across the axis: the contour dips below the pole

Next: module 6a: Landau's initial-value problem for Langmuir waves.