

Plasma Physics and Dynamo Theory • Module 5b

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# The Vlasov–Maxwell system

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# Outline

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1. Properties of the Vlasov–Maxwell system
2. Equilibria: the Jeans theorem
3. Moments
4. Cold-beam instabilities

## Reading

Nicholson, ch. 4–5  
Krall & Trivelpiece, ch. 7  
Kulsrud, ch. 7

**Notes:** chapter 5, §5.3–5.5

# Where we are

## Key result: recap

- ▶ Klimontovich: exact; averaging gives mean fields plus correlations (collisions)
- ▶ BBGKY:  $f_2 = f_1 f_1 + g$ , with  $g/f_1 f_1 \sim 1/N_D$
- ▶ Vlasov: the limit  $N_D \rightarrow \infty$ ;  $f$  is constant along orbits in the self-consistent fields
- ▶ today: what the Vlasov–Maxwell system conserves, its equilibria, its moments, and its simplest instability

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s} \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0, \quad \rho = \sum_s q_s \int f_s d^3 v, \quad \mathbf{j} = \sum_s q_s \int \mathbf{v} f_s d^3 v$$

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# Properties of the Vlasov– Maxwell system

# Conservation of energy: the quantities

**particles** (species  $s$ )

kinetic energy density

$$\varepsilon_s(\mathbf{x}, t) = \int \frac{1}{2} m_s v^2 f_s d^3 v$$

kinetic energy flux

$$\mathbf{Q}_s(\mathbf{x}, t) = \int \frac{1}{2} m_s v^2 \mathbf{v} f_s d^3 v$$

current density

$$\mathbf{j}_s = q_s \int \mathbf{v} f_s d^3 v, \quad \mathbf{j} = \sum_s \mathbf{j}_s$$

**fields**

electromagnetic energy density

$$u_{\text{EM}} = \frac{E^2 + B^2}{8\pi}$$

Poynting vector (energy flux)

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}$$

**Plan:** multiply the Vlasov equation of species  $s$  by  $\frac{1}{2} m_s v^2$  and integrate over all velocities, term by term; then add Poynting's theorem for the fields.

## Step 1: time derivative and streaming

Vlasov equation, with the acceleration  $\mathbf{a} = \frac{q_s}{m_s} \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right)$ :

$$\underbrace{\frac{\partial f_s}{\partial t}}_{(i)} + \underbrace{\mathbf{v} \cdot \nabla f_s}_{(ii)} + \underbrace{\mathbf{a} \cdot \frac{\partial f_s}{\partial \mathbf{v}}}_{(iii)} = 0 \quad \Bigg| \times \frac{1}{2} m_s v^2, \int d^3 v$$

**(i)**  $t$  and  $\mathbf{v}$  are independent variables, so  $\partial_t$  comes out of the velocity integral:

$$\int \frac{1}{2} m_s v^2 \frac{\partial f_s}{\partial t} d^3 v = \frac{\partial}{\partial t} \int \frac{1}{2} m_s v^2 f_s d^3 v = \frac{\partial \varepsilon_s}{\partial t}$$

**(ii)**  $\mathbf{x}$  and  $\mathbf{v}$  are independent too, so  $\mathbf{v} \cdot \nabla f_s = \nabla \cdot (\mathbf{v} f_s)$  and  $\nabla$  comes out:

$$\int \frac{1}{2} m_s v^2 \mathbf{v} \cdot \nabla f_s d^3 v = \nabla \cdot \int \frac{1}{2} m_s v^2 \mathbf{v} f_s d^3 v = \nabla \cdot \mathbf{Q}_s$$

## Step 2: the force term

In components (sum over repeated indices  $k = x, y, z$ ), integrate by parts in  $v_k$ :

$$\int \frac{1}{2} m_s v^2 a_k \frac{\partial f_s}{\partial v_k} d^3 v = \underbrace{\left[ \frac{1}{2} m_s v^2 a_k f_s \right]_{v_k = -\infty}^{+\infty}}_{=0: f_s \rightarrow 0 \text{ faster than any power}} - \int f_s \frac{\partial}{\partial v_k} \left( \frac{1}{2} m_s v^2 a_k \right) d^3 v$$

Product rule, with  $\partial(v^2)/\partial v_k = 2v_k$ :

$$\frac{\partial}{\partial v_k} \left( \frac{1}{2} m_s v^2 a_k \right) = m_s v_k a_k + \frac{1}{2} m_s v^2 \frac{\partial a_k}{\partial v_k}$$

$\partial a_k / \partial v_k = 0$ :  $\mathbf{E}$  does not depend on  $\mathbf{v}$ ;

$(\mathbf{v} \times \mathbf{B})_k = \epsilon_{klm} v_l B_m$ , so

$\partial_{v_k} (\epsilon_{klm} v_l B_m) = \epsilon_{kk m} B_m = 0$  ( $\epsilon$  antisymmetric).

$\mathbf{v} \cdot \mathbf{a}$ : since  $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0$ ,

$$m_s \mathbf{v} \cdot \mathbf{a} = q_s \mathbf{v} \cdot \mathbf{E}$$

The magnetic force does no work.

## Step 2 (cont.): energy equation for one species

Putting the pieces together, the force term is

$$-\int f_s m_s \mathbf{v} \cdot \mathbf{a} d^3v = -q_s \mathbf{E} \cdot \int \mathbf{v} f_s d^3v = -\mathbf{E} \cdot \mathbf{j}_s$$

( $\mathbf{E}$  depends on  $\mathbf{x}$  and  $t$  only, so it comes out of the velocity integral). Collecting (i), (ii), (iii):

### Key result: kinetic energy of species $s$

$$\frac{\partial \varepsilon_s}{\partial t} + \nabla \cdot \mathbf{Q}_s = \mathbf{j}_s \cdot \mathbf{E}$$

- ▶ local conservation law: density  $\varepsilon_s$ , flux  $\mathbf{Q}_s$ , source  $\mathbf{j}_s \cdot \mathbf{E}$
- ▶  $\mathbf{j}_s \cdot \mathbf{E}$  is the power per unit volume done by the electric field on species  $s$  (Joule work); it can have either sign
- ▶ no magnetic term:  $\mathbf{B}$  only bends orbits

## Step 3: the field energy (Poynting's theorem)

Use Faraday,  $\partial_t \mathbf{B} = -c \nabla \times \mathbf{E}$ , and Ampère–Maxwell,  $\partial_t \mathbf{E} = c \nabla \times \mathbf{B} - 4\pi \mathbf{j}$ :

$$\frac{\partial}{\partial t} \frac{E^2}{8\pi} = \frac{\mathbf{E}}{4\pi} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{c}{4\pi} \mathbf{E} \cdot \nabla \times \mathbf{B} - \mathbf{j} \cdot \mathbf{E}, \quad \frac{\partial}{\partial t} \frac{B^2}{8\pi} = \frac{\mathbf{B}}{4\pi} \cdot \frac{\partial \mathbf{B}}{\partial t} = -\frac{c}{4\pi} \mathbf{B} \cdot \nabla \times \mathbf{E}$$

Add them and use the vector identity  $\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{B}$ :

$$\frac{\partial u_{\text{EM}}}{\partial t} = -\frac{c}{4\pi} \nabla \cdot (\mathbf{E} \times \mathbf{B}) - \mathbf{j} \cdot \mathbf{E}$$

### Key result: Poynting's theorem

$$\frac{\partial u_{\text{EM}}}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j} \cdot \mathbf{E}, \quad u_{\text{EM}} = \frac{E^2 + B^2}{8\pi}, \quad \mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}$$

The same  $\mathbf{j} \cdot \mathbf{E}$ , with the opposite sign: what the particles gain, the field loses.

## Step 4: total energy is conserved

Sum the species equations and add Poynting: the  $\mathbf{j} \cdot \mathbf{E}$  terms cancel,

$$\frac{\partial}{\partial t} \left( \sum_s \varepsilon_s + u_{\text{EM}} \right) + \nabla \cdot \left( \sum_s \mathbf{Q}_s + \mathbf{S} \right) = 0$$

Integrate over a volume  $V$ : the flux becomes a surface integral (divergence theorem), zero for a periodic box or if particles and fields vanish at the boundary:

### Key result

$$\frac{d}{dt} \left[ \sum_s \int_V \int \frac{1}{2} m_s v^2 f_s d^3x d^3v + \int_V \frac{E^2 + B^2}{8\pi} d^3x \right] = 0$$

Landau damping (ch. 6): field  $\rightarrow$  particles,  $\mathbf{j} \cdot \mathbf{E} > 0$ ; two-stream: beams  $\rightarrow$  field,  $\mathbf{j} \cdot \mathbf{E} < 0$ . Multiplying by 1 or by  $m_s \mathbf{v}$  instead gives conservation of particle number and of momentum (particles plus field momentum  $\mathbf{E} \times \mathbf{B}/4\pi c$ ).

# Infinitely many invariants: Casimirs and entropy

For any function  $G$ , consider  $C_G = \iint G(f) d^3x d^3v$ . Write Vlasov as  $\partial_t f + \nabla_6 \cdot (\mathbf{V}_6 f) = 0$  with  $\nabla_6 \cdot \mathbf{V}_6 = 0$ ,  $\mathbf{V}_6 = (\mathbf{v}, \mathbf{a})$ :

$$\frac{dC_G}{dt} = \int G'(f) \frac{\partial f}{\partial t} = - \int G'(f) \mathbf{V}_6 \cdot \nabla_6 f = - \int \mathbf{V}_6 \cdot \nabla_6 G(f) = - \int \nabla_6 \cdot (\mathbf{V}_6 G(f)) = 0$$

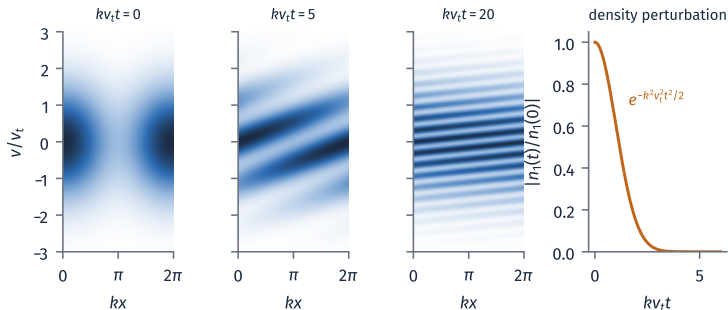
## Key result

Every  $C_G$  is conserved, e.g.  $\int f^2$ , and the entropy  $S = - \iint f \ln f d^3x d^3v$ .

- ▶ the volume of phase space with  $f$  above any given value is constant
- ▶ the entropy cannot grow: **a collisionless plasma does not relax to a Maxwellian**
- ▶ yet perturbations do decay (Landau damping). How?

# Time reversibility and phase mixing

Vlasov–Maxwell is invariant under  $t \rightarrow -t$ ,  $\mathbf{v} \rightarrow -\mathbf{v}$ ,  $\mathbf{B} \rightarrow -\mathbf{B}$ . Simplest case: free streaming,  $\partial_t f + v \partial_x f = 0$ , solved by  $f = f_0(x - vt, v)$ . Start from  $f = f_M(v) [1 + \epsilon \cos kx]$ :



$n_1(t) \propto \int f_M(v) e^{ik(x-vt)} dv = e^{ikx} e^{-k^2 v_t^2 t^2 / 2}$ : the density decays,  $f$  does not. The information moves to ever finer velocity scales,  $\Delta v \sim 1/kt$  (phase mixing).

# The plasma echo: the information is still there

Excite  $k_1$  at  $t = 0$ ; it phase-mixes away:

$$f_1 \propto e^{ik_1(x-vt)}.$$

Excite  $k_2$  at  $t = \tau$ . The second-order term contains

$$e^{ik_2[x-v(t-\tau)]} e^{-ik_1(x-vt)} = e^{i(k_2-k_1)x} e^{-iv[k_2(t-\tau)-k_1t]}$$

The velocity phase vanishes, and a density perturbation **reappears**, at

$$t_{\text{echo}} = \frac{k_2}{k_2 - k_1} \tau$$

## Historical note

Predicted by Gould, O'Neil & Malmberg (1967); measured the next year by Malmberg, Wharton, Gould & O'Neil in a cylindrical plasma at General Atomic (San Diego), exciting waves with two antennas: the echo appeared where theory said.

Phase mixing is reversible; collisions erase the fine structure and make it irreversible.

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# Equilibria: the Jeans theorem

# The Jeans theorem

If  $C_1(\mathbf{x}, \mathbf{v}), \dots, C_k(\mathbf{x}, \mathbf{v})$  are constants of motion in static fields, any  $f = F(C_1, \dots, C_k)$  is a stationary solution of the Vlasov equation:

$$\frac{df}{dt} = \sum_i \frac{\partial F}{\partial C_i} \frac{dC_i}{dt} = 0$$

- ▶ uniform: any  $f(v^2)$ , not only the Maxwellian
- ▶ potential  $\phi$ :  $f(\frac{1}{2}mv^2 + q\phi)$ ; Boltzmann  $n \propto e^{-q\phi/T}$  (ch. 1)
- ▶ mirror (ch. 4):  $f(\mathcal{E}, \mu)$ , e.g. a loss cone

The fields must then come out of  $f$  through Maxwell.



J. H. Jeans

Jeans (1915): the same equation for stars in a galaxy, the “collisionless Boltzmann equation”.

# A self-consistent equilibrium: the Harris sheet

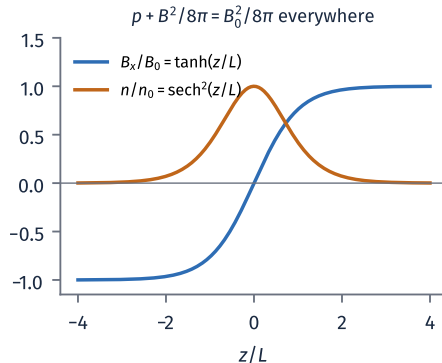
$\mathbf{B} = B_x(z) \hat{\mathbf{x}}$  from  $A_y(z)$ . Constants:  $\mathcal{E}$  and  $p_y = mv_y + qA_y/c$  ( $y$  is ignorable). Take

$$f_s \propto \exp \left[ -\frac{\mathcal{E} - U_s p_y}{T_s} \right]$$

a Maxwellian drifting along  $y$  at  $U_s$ , with  $n_s \propto e^{q_s U_s A_y / c T_s}$ . Ampère,  $d^2 A_y / dz^2 = -(4\pi/c) j_y(A_y)$ , then gives

$$B_x = B_0 \tanh \frac{z}{L}, \quad n = n_0 \operatorname{sech}^2 \frac{z}{L}$$

with  $n_0(T_e + T_i) = B_0^2 / 8\pi$ .



Harris (1962): model of the magnetotail current sheet crossed by Geotail, Cluster, MMS; starting point of reconnection (ch. 15).

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# Moments

# Velocity moments

$$n = \int f \, d^3v, \quad n\mathbf{u} = \int \mathbf{v} f \, d^3v, \quad P = m \int (\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u}) f \, d^3v, \quad \mathbf{q} = \frac{m}{2} \int |\mathbf{v}-\mathbf{u}|^2 (\mathbf{v}-\mathbf{u}) f \, d^3v$$

Maxwellian:  $P = nT \mathbf{I}$ ,  $\mathbf{q} = 0$ . Integrate Vlasov over  $\mathbf{v}$ ; the force term is a divergence, as  $\nabla_{\mathbf{v}} \cdot \mathbf{a} = 0$ :

$$\int \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} \, d^3v = \int \nabla_{\mathbf{v}} \cdot (\mathbf{a} f) \, d^3v = 0 \quad \Rightarrow \quad \frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0$$

Multiply by  $m\mathbf{v}$  first:  $\int m v_j \partial_{v_k} (a_k f) = -m \int a_j f$ , and  $m \int \mathbf{v} \mathbf{v} f = mn\mathbf{u}\mathbf{u} + P$ :

## Key result

$$mn \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla \cdot P + qn \left( \mathbf{E} + \frac{\mathbf{u} \times \mathbf{B}}{c} \right)$$

**Closure:**  $n$  needs  $\mathbf{u}$ ,  $\mathbf{u}$  needs  $P$ ,  $P$  needs  $\mathbf{q}$ ... (chapters 9–10).

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# Cold-beam instabilities

# Linear waves from Vlasov

Electrons, 1D, ions fixed:  $f = f_0(v) + f_1$ ,  $f_1, E \propto e^{i(kx - \omega t)}$ . Linearise:

$$\frac{\partial f_1}{\partial t} + v \frac{\partial f_1}{\partial x} - \frac{e}{m} E \frac{df_0}{dv} = 0 \Rightarrow f_1 = \frac{ie}{m} \frac{E f'_0(v)}{\omega - kv}$$

Poisson,  $ikE = -4\pi e \int f_1 dv$ :

**Key result: electrostatic dispersion relation**

$$D(k, \omega) = 1 + \frac{4\pi e^2}{mk} \int \frac{f'_0(v)}{\omega - kv} dv = 0$$

- ▶ the denominator vanishes at  $v = \omega/k$ : **resonant particles**. What does the integral mean? Next modules: a mathematical interlude, then Landau damping
- ▶ no problem if no particles sit at  $v = \omega/k$ : cold beams

# Two counter-streaming cold beams

$f_0 = \frac{n}{2} [\delta(v - v_0) + \delta(v + v_0)]$ . By parts,  
 $\int \frac{f_0'}{\omega - kv} dv = -k \int \frac{f_0}{(\omega - kv)^2} dv$ :

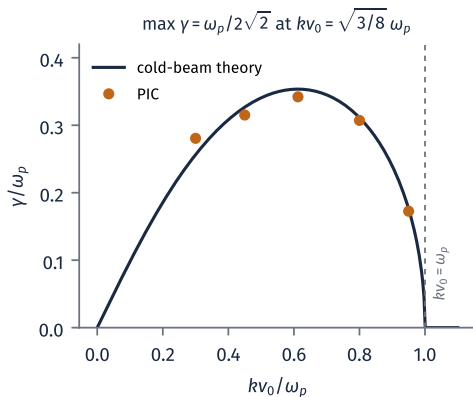
$$1 = \frac{\omega_p^2}{2} \left[ \frac{1}{(\omega - kv_0)^2} + \frac{1}{(\omega + kv_0)^2} \right]$$

Common denominator:

$(\omega^2 - k^2 v_0^2)^2 = \omega_p^2 (\omega^2 + k^2 v_0^2)$ , a quadratic in  $\omega^2$ :

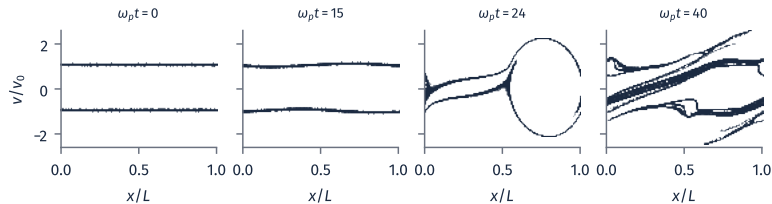
$$\omega^2 = k^2 v_0^2 + \frac{1}{2} \omega_p^2 \pm \frac{1}{2} \omega_p \sqrt{\omega_p^2 + 8k^2 v_0^2}$$

Minus root  $< 0$  iff  $kv_0 < \omega_p$ :  $\omega = \pm i\gamma$ , growth;  
 maximum  $\gamma_{\max} = \omega_p/2\sqrt{2}$  at  $kv_0 = \sqrt{3/8} \omega_p$ .

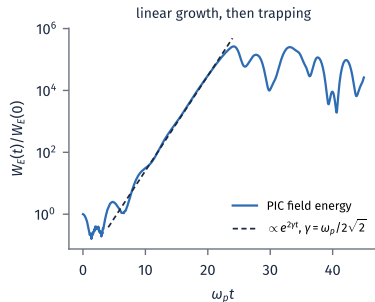


Points: growth rates measured in a 1D particle-in-cell simulation ( $10^5$  electrons).

# Beyond linear theory: trapping and electron holes



- ▶ linear phase: beams ripple, field energy grows as  $e^{2\gamma t}$
- ▶ saturation when electrons are **trapped** in the wave: bounce frequency  $\omega_b = (ekE/m)^{1/2} \sim \gamma$
- ▶ a phase-space vortex survives: an **electron hole**, an exact nonlinear Vlasov equilibrium (Bernstein, Greene & Kruskal 1957)



# Electron holes in space



Geotail. NASA

**Geotail** (ISAS/NASA, 1992–2022) in the magnetotail: “broadband electrostatic noise” resolved into trains of isolated bipolar pulses of  $E_{\parallel}$ , **electrostatic solitary waves** (Matsumoto et al. 1994)

- ▶ interpreted as electron phase-space holes moving along  $\mathbf{B}$
- ▶ later found by Polar, FAST, Cluster, MMS, Parker Solar Probe: wherever electron beams flow

## Historical note

Two-stream instability: Pierce (1948), Haeff (1949) and Bohm & Gross (1949), first in electron tubes; the electron–ion version is the Buneman (1958) instability.

## Check your understanding

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1. The entropy  $-\int f \ln f$  is exactly conserved, yet density perturbations decay. Where does the perturbation go?
2. What is a plasma echo, and why does its observation prove that phase mixing is reversible?
3. Is the Maxwellian the only stationary solution of the Vlasov equation? What singles it out?
4. In the Harris sheet, what balances the magnetic pressure of the lobes, and what carries the current?
5. Why do the moment equations never close by themselves?
6. In the two-stream instability, why are only waves with  $k v_0 < \omega_p$  unstable? What stops the growth?

Exercises 11–18 in chapter 5 of the notes.

# So far, and next

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## In this module

- ▶ energy, momentum, particles, and every  $\int G(f)$  conserved; entropy does not grow
- ▶ reversible, but phase mixing hides perturbations at fine  $\Delta v$ ; the echo brings them back
- ▶ Jeans:  $f$  (constants of motion) is an equilibrium; Harris sheet
- ▶ moments: an unclosed chain  
 $n \rightarrow u \rightarrow P \rightarrow q$
- ▶ cold beams:  $\gamma_{\max} = \omega_p/2\sqrt{2}$ , saturation by trapping

**Next:** what does  $\int f'_0/(\omega - kv) dv$  mean when  $f'_0(\omega/k) \neq 0$ ? A mathematical module on distributions and analytic continuation, then Landau damping (chapter 6).