Plasma Physics and Dynamo Theory • Module 5a

From particles to the Vlasov equation

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Outline

1. The microscopic description
2. The BBGKY hierarchy and the Vlasov limit
3. Application: radiation on the way to Mars

Reading

Nicholson, ch. 3–4
Krall & Trivelpiece, ch. 7
Fitzpatrick, ch. 3

Notes: chapter 5, §5.1–5.2

Why a kinetic description?

Chapters 3–4: one particle in given fields. Chapter 2: collisions as binary events. Now: many particles and the fields they make themselves.

Fluids are not enough when

- the mean free path exceeds the scale of interest
- particles resonate with waves ($v = \omega/k$)
- beams, loss cones, power-law tails: f far from Maxwellian

Solar wind at 1 au

$n_e = 5\ \mathrm{cm}^{-3}$, $T = 10^5$ K:

$$\nu_{ei} \approx 1.4\times10^{-5}\ \mathrm{s}^{-1}, \quad v_{te} \approx 1.2\times10^{8}\ \mathrm{cm\,s}^{-1}$$

$$\lambda_{\mathrm{mfp}} = v_{te}/\nu_{ei} \approx 0.6\ \mathrm{au}$$

comparable to the distance from the Sun.

Levels: N orbits → Klimontovich (exact) → Vlasov + collisions → fluid moments → MHD. Today: the first two arrows.

The distribution function

For each species, $f(\boldsymbol{x}, \boldsymbol{v}, t)\, \mathrm{d}^3x\, \mathrm{d}^3v$ = number of particles in the phase-space cell $\mathrm{d}^3x\, \mathrm{d}^3v$ around $(\boldsymbol{x}, \boldsymbol{v})$.

$$N = \int \mathrm{d}^3x \int \mathrm{d}^3v\, f$$

$$n(\boldsymbol{x}, t) = \int \mathrm{d}^3v\, f, \qquad n\boldsymbol{u} = \int \mathrm{d}^3v\, \boldsymbol{v} f$$

Example: the Maxwellian

$$f_M = n \left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{m|\boldsymbol{v} - \boldsymbol{u}|^2}{2T}\right)$$

Questions for today:

- what equation does f obey, and where does it come from?
- what is lost in going from N particles to a smooth f?

Measuring f in space

NASA/Chris Gunn

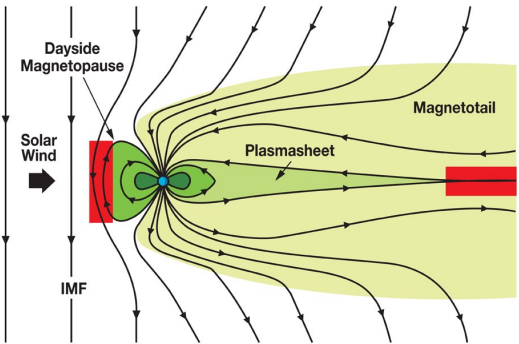
Magnetospheric Multiscale (MMS), NASA, launched 2015

- four identical spacecraft (here stacked for testing), flying in a tetrahedron a few km apart
- the Fast Plasma Investigation counts electrons by energy and direction: a full 3D $f_e(\boldsymbol{v})$ every 30 ms, ions every 150 ms
- measured f are not Maxwellian: beams, crescents in reconnection sites, loss cones

Similar instruments: Parker Solar Probe (SWEAP), Solar Orbiter (SWA), Cluster.

In space plasmas f is an observable, not only a theoretical device.

MMS: where, and why there



NASA

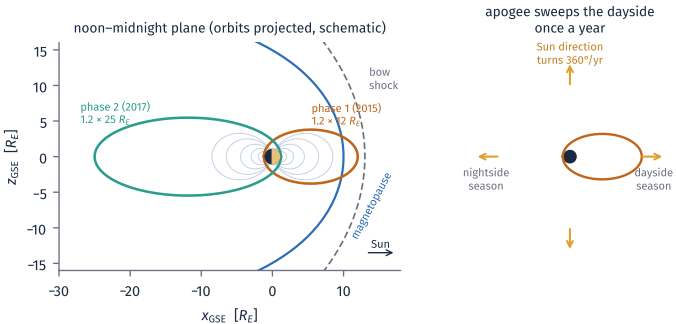
Target: magnetic reconnection (chapter 15), at two sites (red):

- dayside magnetopause, where the solar-wind field meets the Earth's
- magnetotail current sheet (a Harris sheet, module 5b)

The electron diffusion region is $\sim d_e \approx 2$ km thick and sweeps past in $\lesssim 0.1$ s: f_e is needed every 30 ms.

Crossed on the dayside on 16 Oct 2015 (Burch et al. 2016), in the tail on 11 Jul 2017 (Torbert et al. 2018).

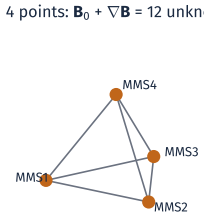
MMS: the orbits

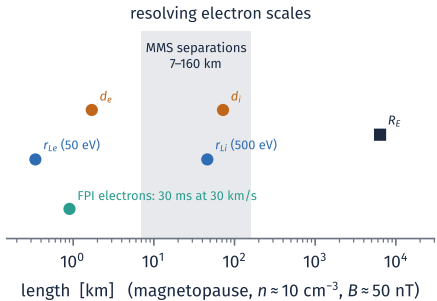


Phase 1 (2015–17): perigee 1.2 R_E, apogee 12 R_E, inclination 28°; semi-major axis $a = 6.6\,R_E$, so period 24 h, like a geostationary orbit. **Phase 2** (from 2017): apogee 25 R_E, $a = 13.1\,R_E$, period $24\,\text{h} \times (13.1/6.6)^{3/2} \approx 2.8$ days (Kepler).

The ellipse is fixed in space while the Sun direction turns once a year: the apogee scans the dayside in one season and the tail six months later.

Why four spacecraft?

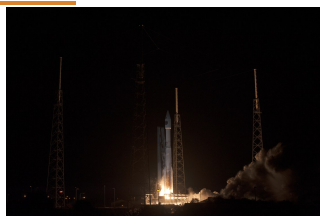




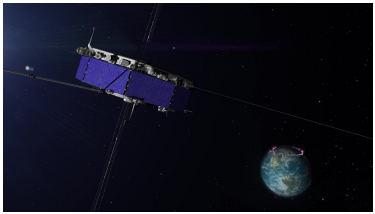
One spacecraft cannot tell space from time. Across the tetrahedron $\boldsymbol{B}(\boldsymbol{r}) \simeq \boldsymbol{B}_0 + (\boldsymbol{r} - \boldsymbol{r}_0) \cdot \nabla \boldsymbol{B}$: $3 + 9 = 12$ unknowns, and four spacecraft $\times$ 3 components = 12 equations.

Then Ampère gives the current, the curlometer: $\boldsymbol{j} = (c/4\pi)\nabla \times \boldsymbol{B}$, with $\nabla \cdot \boldsymbol{B} = 0$ as a check. Separations of 7–160 km straddle d_e and d_i.

The spacecraft



Atlas V, Cape Canaveral, 12 March 2015. NASA



artist's view. NASA Goddard

- octagonal, 3.5 m across, $\approx$ 1360 kg each; all four on one rocket
- spin-stabilised, one turn every 20 s
- four 60 m wire booms in the spin plane, two 12.5 m axial booms: $\boldsymbol{E}$ from the voltage between probes
- magnetometers on 5 m booms, away from the spacecraft's own fields
- FPI: eight electron and eight ion spectrometer heads per spacecraft see all directions at once: $f(\boldsymbol{v})$ in 30 ms, not one 20 s spin

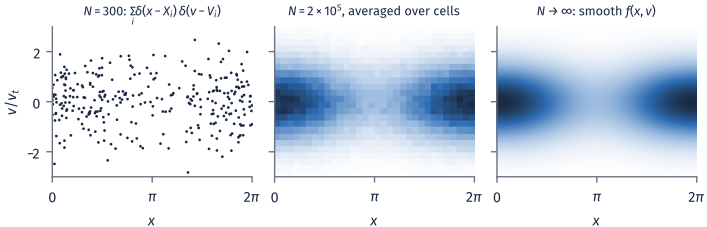
The microscopic description

The Klimontovich density

N point particles with orbits $\boldsymbol{X}_i(t)$, $\boldsymbol{V}_i(t)$. Their exact phase-space density:

Key result

$$\mathcal{N}(\boldsymbol{x}, \boldsymbol{v}, t) = \sum_{i=1}^{N} \delta\big(\boldsymbol{x} - \boldsymbol{X}_i(t)\big)\, \delta\big(\boldsymbol{v} - \boldsymbol{V}_i(t)\big)$$



$\int \mathcal{N}\, \mathrm{d}^3 v = \sum_i \delta(\boldsymbol{x} - \boldsymbol{X}_i)$ is spiky; a smooth f appears only after averaging.

The Klimontovich equation: derivation

Differentiate one term; only $\boldsymbol{X}_i(t)$, $\boldsymbol{V}_i(t)$ depend on t:

$$\frac{\partial}{\partial t}\,\delta(\boldsymbol{x}-\boldsymbol{X}_i)\,\delta(\boldsymbol{v}-\boldsymbol{V}_i) = -\left[\dot{\boldsymbol{X}}_i\cdot\nabla_{\boldsymbol{x}} + \dot{\boldsymbol{V}}_i\cdot\nabla_{\boldsymbol{v}}\right]\delta(\boldsymbol{x}-\boldsymbol{X}_i)\,\delta(\boldsymbol{v}-\boldsymbol{V}_i)$$

Equations of motion in the microscopic fields $\boldsymbol{E}^m$, $\boldsymbol{B}^m$ of all particles:

$$\dot{\boldsymbol{X}}_i = \boldsymbol{V}_i, \qquad \dot{\boldsymbol{V}}_i = \frac{q}{m}\left[\boldsymbol{E}^m(\boldsymbol{X}_i,t) + \frac{\boldsymbol{V}_i\times\boldsymbol{B}^m(\boldsymbol{X}_i,t)}{c}\right]$$

Inside the δ functions $\boldsymbol{X}_i\to\boldsymbol{x}$, $\boldsymbol{V}_i\to\boldsymbol{v}$; sum over i:

Key result: Klimontovich equation

$$\frac{\partial\mathcal{N}}{\partial t} + \boldsymbol{v}\cdot\nabla\mathcal{N} + \frac{q}{m}\left(\boldsymbol{E}^m + \frac{\boldsymbol{v}\times\boldsymbol{B}^m}{c}\right)\cdot\frac{\partial\mathcal{N}}{\partial\boldsymbol{v}} = 0$$

with Maxwell's equations sourced by $\rho^m = \sum_s q_s\int\mathcal{N}_s\,\mathrm{d}^3v$, $\boldsymbol{j}^m = \sum_s q_s\int\boldsymbol{v}\,\mathcal{N}_s\,\mathrm{d}^3v$.

Exact, and useless as it stands

No approximation: equivalent to Newton + Maxwell for all N particles, so solving it means following every orbit, from $6N$ initial conditions we never know. **Average** over an ensemble of systems with the same macroscopic state:

$$\mathcal{N} = f + \delta\mathcal{N}, \quad \boldsymbol{E}^m = \boldsymbol{E} + \delta\boldsymbol{E}, \quad \boldsymbol{B}^m = \boldsymbol{B} + \delta\boldsymbol{B}, \qquad f = \langle \mathcal{N} \rangle\,,\ \langle \delta\mathcal{N} \rangle = 0$$

Only the force term is nonlinear: $\langle \boldsymbol{E}^m \cdot \partial_{\boldsymbol{v}} \mathcal{N} \rangle = \boldsymbol{E} \cdot \partial_{\boldsymbol{v}} f + \langle \delta\boldsymbol{E} \cdot \partial_{\boldsymbol{v}} \delta\mathcal{N} \rangle$, so

Key result

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m}\left(\boldsymbol{E} + \frac{\boldsymbol{v} \times \boldsymbol{B}}{c}\right) \cdot \frac{\partial f}{\partial \boldsymbol{v}} = -\frac{q}{m}\left\langle \left(\delta\boldsymbol{E} + \frac{\boldsymbol{v} \times \delta\boldsymbol{B}}{c}\right) \cdot \frac{\partial \delta\mathcal{N}}{\partial \boldsymbol{v}} \right\rangle \equiv \left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}}$$

Left: smooth f in smooth fields. Right: correlated fluctuations = collisions.

Liouville's theorem in $6N$ dimensions

A point $Z = (\boldsymbol{x}_1, \boldsymbol{v}_1, \ldots, \boldsymbol{x}_N, \boldsymbol{v}_N)$ in $6N$ dimensions; $f_N(Z,t)$ = probability density of the whole system. Probability is conserved:

$$\frac{\partial f_N}{\partial t} + \nabla_Z \cdot (\dot{Z} f_N) = 0, \qquad \nabla_Z \cdot \dot{Z} = \sum_i \Big(\underbrace{\nabla_{\boldsymbol{x}_i} \cdot \boldsymbol{v}_i}_{=0} + \nabla_{\boldsymbol{v}_i} \cdot \boldsymbol{a}_i \Big)$$

$\boldsymbol{a}_i = (q/m)(\boldsymbol{E}^m + \boldsymbol{v}_i \times \boldsymbol{B}^m/c)$; $\boldsymbol{E}^m$ does not depend on $\boldsymbol{v}_i$, and $\partial_{v_x}(v_y B_z - v_z B_y) + \partial_{v_y}(v_z B_x - v_x B_z) + \partial_{v_z}(v_x B_y - v_y B_x) = 0$.

Key result: Liouville

$\nabla_Z \cdot \dot{Z} = 0$: the phase-space flow is incompressible, and f_N is constant along trajectories, $\mathrm{d}f_N/\mathrm{d}t = \partial_t f_N + \dot{Z} \cdot \nabla_Z f_N = 0$.

The BBGKY hierarchy and the Vlasov limit

Reduced distributions

One species, electrostatic pair force $\boldsymbol{F}_{ij} = q^2(\boldsymbol{x}_i - \boldsymbol{x}_j)/|\boldsymbol{x}_i - \boldsymbol{x}_j|^3$ (plus a neutralising background). Write $i \equiv (\boldsymbol{x}_i, \boldsymbol{v}_i)$:

$$\frac{\partial f_N}{\partial t} + \sum_i \boldsymbol{v}_i \cdot \frac{\partial f_N}{\partial \boldsymbol{x}_i} + \sum_i \sum_{j \neq i} \frac{\boldsymbol{F}_{ij}}{m} \cdot \frac{\partial f_N}{\partial \boldsymbol{v}_i} = 0$$

f_N symmetric in the particles, $\int f_N \, \mathrm{d}1 \cdots \mathrm{d}N = 1$. Reduced distributions:

$$f_1(1) = N \int f_N \, \mathrm{d}2 \cdots \mathrm{d}N, \qquad f_2(1,2) = N(N-1) \int f_N \, \mathrm{d}3 \cdots \mathrm{d}N$$

- $f_1(\boldsymbol{x}, \boldsymbol{v}, t)$ is the distribution function f: $\int f_1 \, \mathrm{d}^3 v = n$
- $f_2(1,2)$: density of *pairs*, one at 1 and one at 2
- uncorrelated particles: $f_2(1,2) = f_1(1) f_1(2)$

Integrating Liouville over particles $2 \dots N$

Multiply by N and integrate over $\mathrm{d}2 \cdots \mathrm{d}N$, term by term:

- ∂_t and $\boldsymbol{v}_1 \cdot \partial_{\boldsymbol{x}_1}$: give $\partial_t f_1 + \boldsymbol{v}_1 \cdot \partial_{\boldsymbol{x}_1} f_1$
- $\boldsymbol{v}_i \cdot \partial_{\boldsymbol{x}_i} f_N$, $i \geq 2$: a divergence in $\boldsymbol{x}_i$, integrates to zero
- $\boldsymbol{F}_{ij} \cdot \partial_{\boldsymbol{v}_i} f_N$, $i \geq 2$: a divergence in $\boldsymbol{v}_i$ ($\boldsymbol{F}_{ij}$ independent of $\boldsymbol{v}_i$), zero
- $i = 1$: $N - 1$ identical terms $j = 2, \dots, N$:

$$N(N-1) \int \frac{\boldsymbol{F}_{12}}{m} \cdot \frac{\partial}{\partial \boldsymbol{v}_1} \left[\int f_N \, \mathrm{d}3 \cdots \mathrm{d}N \right] \mathrm{d}2 = \int \frac{\boldsymbol{F}_{12}}{m} \cdot \frac{\partial f_2(1,2)}{\partial \boldsymbol{v}_1} \, \mathrm{d}2$$

Key result: first BBGKY equation

$$\frac{\partial f_1}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial f_1}{\partial \boldsymbol{x}_1} + \int \frac{\boldsymbol{F}_{12}}{m} \cdot \frac{\partial f_2(1,2)}{\partial \boldsymbol{v}_1} \, \mathrm{d}2 = 0$$

The equation for f_2 contains f_3, and so on: a hierarchy (Bogoliubov, Born, Green, Kirkwood, Yvon).

Closing the hierarchy: mean field plus correlations

Split the pair distribution:

$$f_2(1,2) = f_1(1)\, f_1(2) + g(1,2)$$

Uncorrelated part: the force from all other particles, smoothed,

$$\int \boldsymbol{F}_{12}\, f_1(2)\, \mathrm{d}2 = q^2 \int \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|^3}\, n(\boldsymbol{x}_2)\, \mathrm{d}^3 x_2 = q\, \boldsymbol{E}(\boldsymbol{x}_1)$$

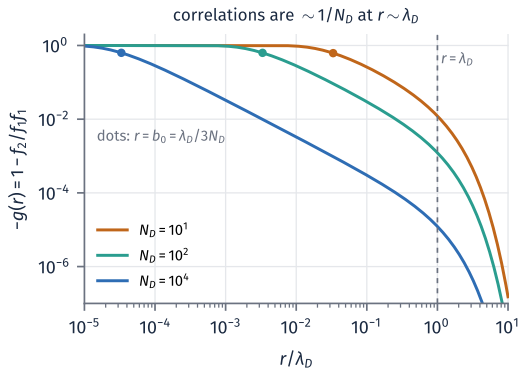
the Coulomb field of the smooth charge density: Poisson's equation.

Key result

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} + \frac{q}{m} \boldsymbol{E} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = -\int \frac{\boldsymbol{F}_{12}}{m} \cdot \frac{\partial g(1,2)}{\partial \boldsymbol{v}_1}\, \mathrm{d}2$$

Same structure as the averaged Klimontovich equation: mean field on the left, correlations ($g \leftrightarrow \langle \delta\mathcal{N}\delta\mathcal{N}\rangle$) on the right.

How large are the correlations?



Equilibrium (chapter 1): a particle at 1 sees a Debye potential, so the chance of a neighbour at distance r carries a Boltzmann factor,

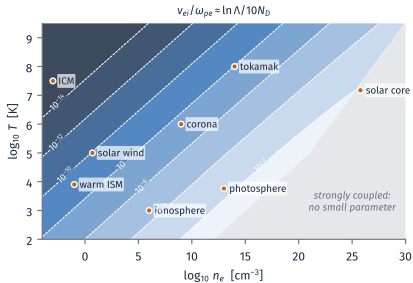
$$\frac{f_2}{f_1 f_1} = \exp\left(-\frac{q\phi(r)}{T}\right), \quad \phi = \frac{q}{r}\, e^{-r/\lambda_D}$$

At $r \sim \lambda_D$:

$$|g|/f_1 f_1 \sim \frac{q^2}{\lambda_D T} = \frac{b_0}{\lambda_D} = \frac{1}{3N_D}$$

Correlations are of order unity only inside b_0, which holds $\sim N_D^{-2}$ particles.

The Vlasov limit



Collision term / mean-field term $\sim \nu_{ei}/\omega_{pe}$ (chapter 2):

$$\frac{\nu_{ei}}{\omega_{pe}} \approx \frac{\ln\Lambda}{10\,N_D}$$

the $1/N_D$ of g, times $\ln\Lambda$ from scales b_0 to λ_D. **Vlasov limit**: $e, m \to 0$, $n \to \infty$ at fixed e/m, ne, T/m: ω_p, λ_D, v_t fixed, $N_D \to \infty$, $g \to 0$.

Key result: Vlasov equation

$$\frac{\partial f_s}{\partial t} + \boldsymbol{v}\cdot\nabla f_s + \frac{q_s}{m_s}\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\frac{\partial f_s}{\partial \boldsymbol{v}} = 0$$

What the Vlasov equation says

Along a particle orbit in the smooth fields, $\dot{\boldsymbol{x}} = \boldsymbol{v}$, $\dot{\boldsymbol{v}} = (q/m)(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}/c)$:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \dot{\boldsymbol{x}} \cdot \nabla f + \dot{\boldsymbol{v}} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0$$

- f is carried unchanged along orbits: phase-space density is an incompressible fluid
- the fields are self-consistent: Maxwell with $\rho = \sum_s q_s \int f_s \, \mathrm{d}^3 v$, $\boldsymbol{j} = \sum_s q_s \int \boldsymbol{v} f_s \, \mathrm{d}^3 v$; the system is nonlinear
- valid for times $t \ll 1/\nu$ and scales $\gg b_0$; collisions add $(\partial f/\partial t)_\mathrm{c}$ (chapter 7)

Solar corona: $\nu_{ei} \approx 70\ \mathrm{s}^{-1}$, $\omega_{pe} \approx 2 \times 10^9\ \mathrm{s}^{-1}$: waves oscillate 10^7 times between collisions. Collisionless on wave time scales, collisional on the scale of a loop's lifetime.

Application: radiation on the way to Mars

Cosmic rays are a collisionless gas

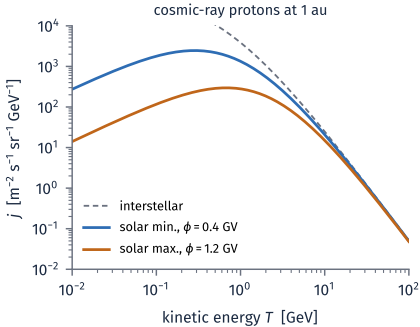
Galactic cosmic rays (GCR): mostly protons, $T \sim 0.1$–10 GeV; mean free path $\gg$ size of the heliosphere. Vlasov and Liouville apply.

Intensity j = number per area, time, solid angle, energy; with $\mathrm{d}^3x = v\,\mathrm{d}t\,\mathrm{d}A$, $\mathrm{d}^3p = p^2\mathrm{d}p\,\mathrm{d}\Omega$, $\mathrm{d}T = v\,\mathrm{d}p$:

$$j\,\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}T = f\,\mathrm{d}^3x\,\mathrm{d}^3p \;\Rightarrow\; j = p^2 f$$

Static $\boldsymbol{B}$: $|\boldsymbol{p}|$ and f conserved $\Rightarrow$ j constant along every trajectory.

Solar-wind electric fields: energy loss Φ on the way in (force field), $j(T)/p^2(T) = j_{\mathrm{LIS}}(T+\Phi)/p^2(T+\Phi)$.



LIS: Burger et al. (2000); force field: Gleeson & Axford (1968)

A trip to Mars: the dose

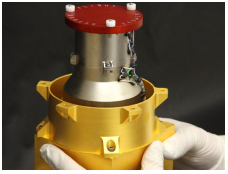
Absorbed dose rate = fluence rate × stopping power, a velocity moment of f:

$$\dot{D} = 4\pi \int j(T)\, \frac{1}{\rho} \left| \frac{\mathrm{d}T}{\mathrm{d}x} \right| \mathrm{d}T$$

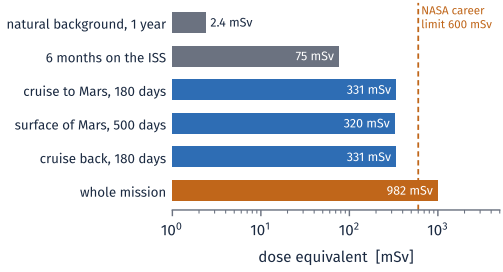
Protons, Bethe: $|\mathrm{d}T/\rho\,\mathrm{d}x| \approx 2.2\ \mathrm{MeV\,cm^2\,g^{-1}}$ at 1 GeV; $4\pi \int j\,\mathrm{d}T \approx 1\text{–}5\ \mathrm{cm^{-2}\,s^{-1}}$: $\dot{D} \approx 0.1\text{–}0.2$ mGy/day. He and heavier nuclei ($\propto Z^2$) double it.

Measured by RAD inside the Curiosity capsule, 2011–12: 0.48 mGy/day, quality factor 3.8: 1.84 mSv/day; on the surface 0.64 mSv/day.

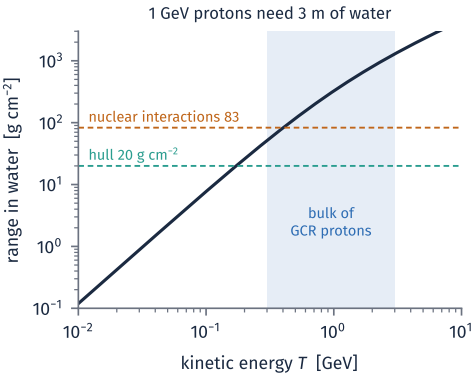
Zeitlin et al. (2013), Hassler et al. (2014)



RAD before flight. NASA/JPL-Caltech/SwRI



Why not shield it?



Matter: range $= \int \mathrm{d}T / |\mathrm{d}T/\rho\,\mathrm{d}x| = 325\ \mathrm{g\,cm^{-2}}$ for 1 GeV (3 m of water); a hull is ~ 20. Nuclei fragment over 83 $\mathrm{g\,cm^{-2}}$: thin shields can add secondaries.

Magnets: $r_L = R/300B$ cm (R in V, B in G): a 2 GeV proton ($R = 2.8$ GV) in 1 T has $r_L \approx 9$ m. Liouville: a field can only make directions inaccessible, never concentrate the flux.

Worse than expected: weak solar cycles 23–24, record GCR fluxes in 2009; the allowed time in deep space is shrinking.

E. N. Parker (2005, 2006): GCR, not solar flares, are the obstacle to human travel to Mars. Solar protons ($\lesssim 100$ MeV, range $\lesssim 8\ \mathrm{g\,cm^{-2}}$) can be stopped in a shelter; GCR cannot.

Check your understanding

1. The Klimontovich equation is exact. Why is it of no direct use, and what does averaging buy?
2. What physical process does the right-hand side of the averaged equation, $-\frac{q}{m}\langle(\delta\boldsymbol{E} + \boldsymbol{v}\times\delta\boldsymbol{B}/c)\cdot\partial_{\boldsymbol{v}}\delta\mathcal{N}\rangle$, describe?
3. The magnetic force depends on $\boldsymbol{v}$. Why does it not spoil Liouville's theorem? Which force would?
4. In which limit is the Vlasov equation exact? Which quantities are kept fixed, and which go to zero?
5. Why can no arrangement of static magnetic fields raise the cosmic-ray intensity above its interplanetary value?
6. Why does a mission like MMS need four spacecraft, and why must f_e be measured every 30 ms?

Exercises 1–10 in chapter 5 of the notes.

So far, and next

In this module

- Klimontovich: exact, a sum of δ's
- averaging splits mean fields from fluctuations; correlated fluctuations are collisions
- Liouville: phase-space flow is incompressible
- BBGKY: f_1 needs f_2; closure $f_2 = f_1 f_1 + g$
- $g \sim 1/N_D$: Vlasov is the limit $N_D \to \infty$, with collisions $\sim \ln\Lambda / N_D$
- cosmic rays: $j = p^2 f$ conserved along orbits; ~ 1 Sv on a Mars mission

Next: module 5b: properties of the Vlasov–Maxwell system, equilibria, moments, and the two-stream instability.

Historical note

Landau's collision integral (1936); Vlasov's equation (1938); BBGKY hierarchy (Yvon 1935; Bogoliubov, Born & Green, Kirkwood 1946); Klimontovich (1967).