

Adiabatic invariants, radiation belts and Fermi acceleration

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Outline

1. Adiabatic invariants as actions
2. The Van Allen radiation belts
3. Fermi acceleration

Reading

Chen, ch. 2

Fitzpatrick, ch. 2

Northrop (1963)

Longair, ch. 17

Notes: chapter 4, §4.3–4.5

Problems: set 2

Where we are

Key result: recap

- ▶ mirror force $-\mu \partial B / \partial s$; $\mu = \frac{1}{2} m v_{\perp}^2 / B$ invariant
- ▶ $\mu B(s)$ acts as a potential: particles bounce between mirrors
- ▶ loss cone $\sin^2 \alpha_{\text{lc}} = B_{\text{min}} / B_{\text{max}}$
- ▶ today: why μ is conserved, two more invariants, and moving mirrors

Adiabatic invariants as actions

The slowly varying oscillator

$H = p^2/2m + \frac{1}{2}m\omega^2(t)x^2$, with $|\dot{\omega}| \ll \omega^2$. Energy not conserved:

$$\frac{dE}{dt} = \frac{\partial H}{\partial t} = \frac{1}{2}m \frac{d\omega^2}{dt} x^2 = m\omega\dot{\omega} x^2$$

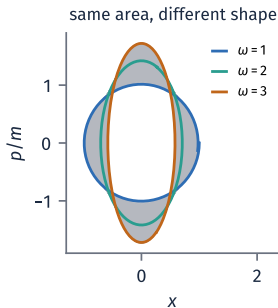
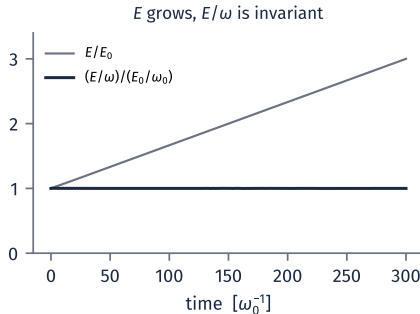
Average over a period: $\frac{1}{2}m\omega^2\langle x^2 \rangle = \frac{1}{2}E$ (virial), so $\langle x^2 \rangle = E/m\omega^2$:

$$\left\langle \frac{dE}{dt} \right\rangle = E \frac{\dot{\omega}}{\omega} \Rightarrow \frac{d}{dt} \left(\frac{E}{\omega} \right) = 0$$

Key result

$E/\omega = J/2\pi$, with $J = \oint p dx$ the **action** (area of the phase-space ellipse).
Gyration: $E_{\perp}/\omega_c = \frac{1}{2}mv_{\perp}^2/\omega_c = (mc/|q|)\mu$.

Phase space: the area is invariant



$\ddot{x} + \omega^2(t)x = 0$, ω growing slowly from 1 to 3 (computed):

- ▶ the orbit is squeezed in x , stretched in p ; its area $J = \oint p dx = 2\pi E/\omega$ stays fixed
- ▶ E triples, E/ω constant to $< 1\%$: for gyration, the invariance of μ

The canonical action of gyration

Canonical momentum $\mathbf{p} = m\mathbf{v} + (q/c)\mathbf{A}$, $\mathbf{B} = \nabla \times \mathbf{A}$. Around the orbit, in the sense of motion:

$$J_1 = \oint \mathbf{p} \cdot d\mathbf{l} = \underbrace{\oint m\mathbf{v}_\perp \cdot d\mathbf{l}}_{2\pi r_L m v_\perp = 2\pi m \omega_c r_L^2} + \underbrace{\frac{q}{c} \int_S \mathbf{B} \cdot d\mathbf{S}}_{\text{Stokes}}$$

Diamagnetic: the flux in the sense of motion is **negative** for both signs,
 $\frac{q}{c} \int \mathbf{B} \cdot d\mathbf{S} = -\frac{|q|}{c} B \pi r_L^2 = -\pi m \omega_c r_L^2$:

$$J_1 = 2\pi m \omega_c r_L^2 - \pi m \omega_c r_L^2 = \pi m \omega_c r_L^2 = \frac{|q|}{c} \Phi = \frac{2\pi m c}{|q|} \mu$$

Careful with the sign: with + one gets $3\pi m \omega_c r_L^2$, still $\propto \mu$ but wrong.

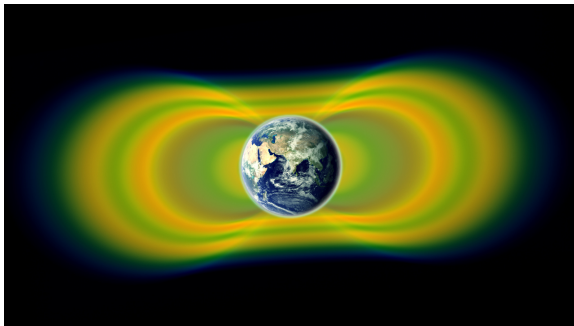
Three invariants

motion	frequency	invariant	broken by changes at
gyration	ω_c	$\mu = \frac{1}{2} m v_{\perp}^2 / B$	$\gtrsim \omega_c$
bounce between mirrors	ω_b	$J = \oint m v_{\parallel} ds$	$\gtrsim \omega_b$
drift around the planet	ω_d	Φ (flux in drift shell)	$\gtrsim \omega_d$

- ▶ $\omega_c \gg \omega_b \gg \omega_d$: invariants progressively more fragile
- ▶ J : path along the same line forth and back encloses no flux, so $\oint A_{\parallel} ds = 0$ and $J = 2 \int_{s_1}^{s_2} m v_{\parallel} ds$
- ▶ μ fixes B at the mirror points, J the length of line between them: together they define a closed **drift shell**
- ▶ Φ : drift so slow that $\oint m v_d dl$ is negligible; action $\propto$ enclosed flux

The Van Allen radiation belts

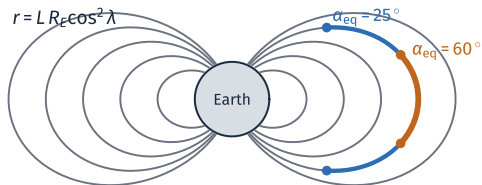
The Van Allen belts



NASA / Van Allen Probes / Goddard Space Flight Center

- ▶ discovered with the Geiger counters of Explorer 1 (1958)
- ▶ **inner belt**, 1–3 R_E : protons > 10 MeV from the decay of neutrons knocked out of the atmosphere by cosmic rays; stable for years
- ▶ **outer belt**, 3–9 R_E : electrons up to several MeV injected from the magnetosphere; varies in hours
- ▶ the Van Allen Probes (2012) saw a transient third belt

The dipole field



A trapped particle gyrates, bounces between mirror points in the two hemispheres, and drifts around the Earth on its L -shell. The geometry is set by the dipole field.

Dipole: $B_r = 2M_E \cos \Theta / r^3$, $B_\Theta = M_E \sin \Theta / r^3$, $B_E = M_E / R_E^3 = 0.31$ G. Field line:
 $dr/B_r = r d\Theta/B_\Theta \Rightarrow dr/r = 2 \cot \Theta d\Theta \Rightarrow r \propto \sin^2 \Theta$:

$$r = LR_E \cos^2 \lambda, \quad B(\lambda) = \frac{B_E}{L^3} \frac{(1 + 3 \sin^2 \lambda)^{1/2}}{\cos^6 \lambda}$$

Mirror points and the loss cone in a dipole

Mirror point of a particle with equatorial pitch angle α_{eq} :

$$\sin^2 \alpha_{\text{eq}} = \frac{B_{\text{eq}}}{B(\lambda_m)} = \frac{\cos^6 \lambda_m}{(1 + 3 \sin^2 \lambda_m)^{1/2}}$$

depends only on α_{eq} , not on L or energy. Lost if it mirrors below the surface. The line reaches $r = R_E$ at $\cos^2 \lambda_E = 1/L$, where $B/B_{\text{eq}} = L^3(4 - 3/L)^{1/2}$:

Key result

$$\sin^2 \alpha_{\text{lc}} = \frac{1}{L^3(4 - 3/L)^{1/2}} = \frac{1}{L^{5/2}(4L - 3)^{1/2}}$$

Geostationary orbit, $L = 6.6$: $\alpha_{\text{lc}} \approx 2.5^\circ$. The loss cone is tiny: almost everything is trapped.

Bounce and drift periods

Bounce: $\tau_b = \oint ds/v_{\parallel} \simeq \frac{4LR_E}{v} (1.30 - 0.56 \sin \alpha_{eq})$ (numerical fit). 1 MeV proton: $\approx 2.4 L$ s; relativistic electron: $\approx 0.1 L$ s.

Drift: equatorial particle ($v_{\parallel} = 0$), only grad- B ; at the equator of a dipole $|\nabla B|/B = 3/r$:

$$v_d = \frac{v_{\perp}^2}{2\omega_c} \frac{3}{r} = \frac{3c\mathcal{E}}{|q|Br} \xrightarrow{B=B_E/L^3, r=LR_E} v_d = \frac{3c\mathcal{E}L^2}{|q|B_ER_E}$$

$$\tau_d = \frac{2\pi LR_E}{v_d} = \frac{2\pi|q|B_ER_E^2}{3c\mathcal{E}L} \approx \frac{0.73}{L} \left(\frac{\mathcal{E}}{1 \text{ MeV}} \right)^{-1} h$$

Ions westward, electrons eastward. $\tau_c \ll \tau_b \ll \tau_d$.

The ring current and magnetic storms

Opposite drifts of ions and electrons: a westward **ring current**, whose field opposes the dipole at the surface.

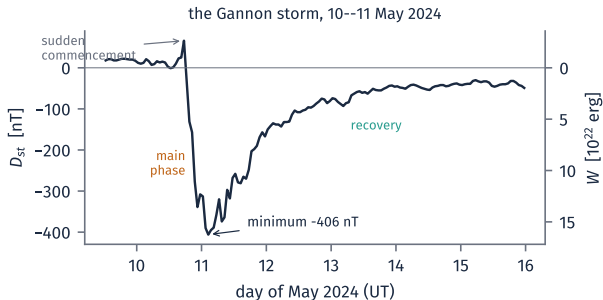
Key result: Dessler–Parker–Sckopke

$$\frac{\Delta B}{B_E} = -\frac{2}{3} \frac{W}{W_M}, \quad W_M = \frac{B_E^2 R_E^3}{3} \approx 8 \times 10^{24} \text{ erg}$$

W : energy of trapped particles; W_M : dipole energy outside the Earth. $\Delta B = -100 \text{ nT}$
 $\Rightarrow W \approx 4 \times 10^{22} \text{ erg}$.

- ▶ D_{st} index: storms driven by CMEs drop it by hundreds of nT in hours
- ▶ slow compression ($\ll \omega_d$): particles conserve Φ , move inwards and gain energy (μ, J conserved)
- ▶ fluctuations at $\sim \omega_d$ break Φ only: **radial diffusion**

A real storm: 10–11 May 2024



data: WDC for Geomagnetism, Kyoto (provisional D_{st} ; Nosé et al. 2015,

doi:10.17593/14515-74000)

- ▶ **sudden commencement:** a CME shock compresses the magnetosphere, $D_{st} > 0$
- ▶ **main phase:** particles injected, ring current grows, $D_{st} \rightarrow -406$ nT in hours
- ▶ **recovery** over days: charge exchange, precipitation
- ▶ right axis: DPS energy, 1.6×10^{23} erg at minimum
- ▶ strongest storm since 2003: aurorae seen from Italy

Fermi acceleration

Reflection from a moving mirror

Fermi (1949): cosmic rays bounce off moving magnetised clouds acting as mirrors. In the mirror frame the energy is unchanged and $v_{\parallel}$ reverses. Back in the lab, mirror speed V :



$$\Delta\mathcal{E} = \frac{1}{2}m \left[(v \pm 2V)^2 - v^2 \right] = \pm 2mvV + 2mV^2$$

Second-order Fermi acceleration

Random mirrors: head-on collisions are more frequent, rate $\propto v + V$ against $v - V$. Rate-weighted average:

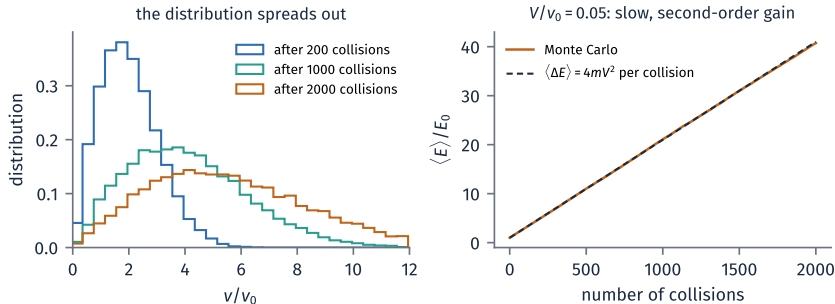
$$\langle \Delta \mathcal{E} \rangle = \frac{(v + V)(2mvV + 2mV^2) + (v - V)(-2mvV + 2mV^2)}{(v + V) + (v - V)} = \frac{8mvV^2}{2v} = 4mV^2$$

Key result

$$\frac{\langle \Delta \mathcal{E} \rangle}{\mathcal{E}} = 8 \frac{V^2}{v^2} \quad (\text{relativistic, 3D: } \frac{8}{3} (V/c)^2)$$

- ▶ **second order**: first-order gains and losses nearly cancel
- ▶ interstellar clouds, $V \sim 10$ km/s: $(V/c)^2 \sim 10^{-9}$, far too slow for cosmic rays
- ▶ efficient in turbulence with fast magnetic fluctuations

Second order in practice: a Monte Carlo experiment



20 000 particles, mirrors with $V = 0.05 v_0$ in random directions, head-on with probability $(v + V)/2v$:

- ▶ $\langle E \rangle$ grows by $4mV^2$ per collision, exactly as derived; 2000 collisions for a factor 40
- ▶ the spread grows faster than the mean: energy **diffuses**, a few particles win big

First-order acceleration and the second invariant

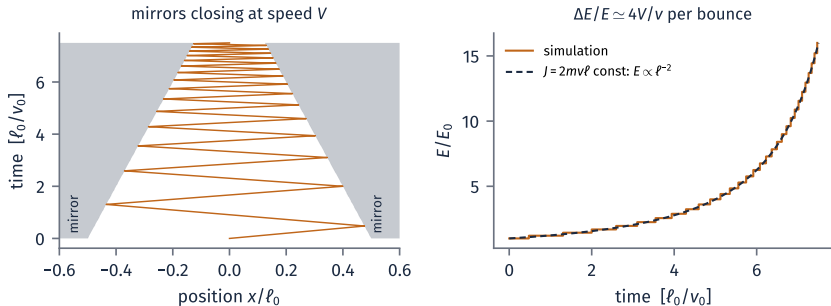
Approaching mirrors: every collision head-on, $\Delta\mathcal{E}/\mathcal{E} \simeq 4V/v$: **first order**. With the second invariant, mirrors a distance ℓ apart closing slowly:

$$J = 2mv_{\parallel}\ell = \text{const} \quad \Rightarrow \quad v_{\parallel} \propto \frac{1}{\ell}, \quad \mathcal{E}_{\parallel} \propto \frac{1}{\ell^2}$$

Key result

Converging flows on the two sides of a **shock** act as approaching mirrors: diffusive shock acceleration (ch. 14), the mechanism for galactic cosmic rays up to $\sim 10^{15}$ eV.

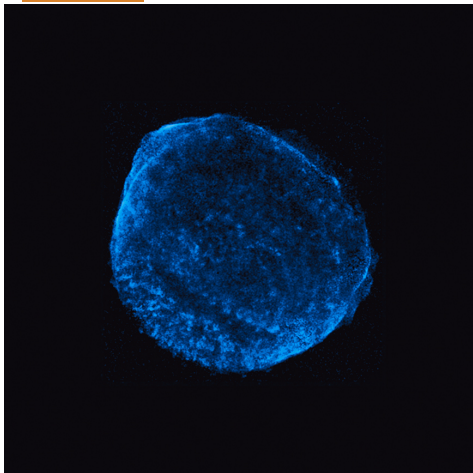
First order in practice: closing mirrors



Two mirrors closing at $V = 0.05 v_0$ each, computed collision by collision:

- ▶ every bounce head-on: a staircase of gains $\Delta E/E \simeq 4V/v$
- ▶ it follows $E \propto \ell^{-2}$ from $J = 2mv\ell$: a factor 16 when $\ell \rightarrow \ell_0/4$

Converging flows in nature: supernova shocks



SN 1006 in X-rays (Chandra), the remnant of the brightest supernova in recorded history.

- ▶ the thin bright rims (NE and SW) are **synchrotron** emission, not thermal
- ▶ X-ray synchrotron in a $\sim 100 \mu\text{G}$ field needs electrons of $\sim 10\text{--}100 \text{ TeV}$
- ▶ they are accelerated at the shock: the upstream and downstream flows converge, like closing mirrors, and particles crossing back and forth gain energy at first order (diffusive shock acceleration, ch. 14)

Why power laws

Each cycle: energy $\times \beta = 1 + \xi$, escape probability P . After k cycles:

$$N = N_0(1 - P)^k, \quad \mathcal{E} = \mathcal{E}_0\beta^k \quad \Rightarrow \quad k = \frac{\ln(\mathcal{E}/\mathcal{E}_0)}{\ln \beta}$$

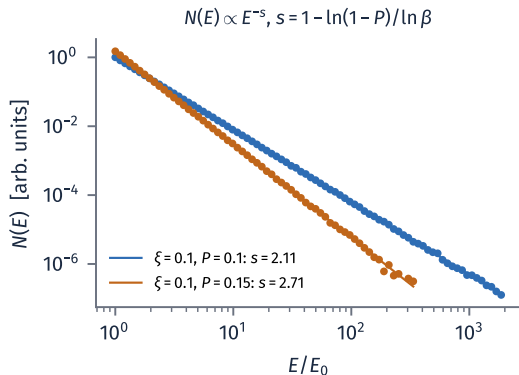
$$\frac{N(> \mathcal{E})}{N_0} = (1 - P)^{\ln(\mathcal{E}/\mathcal{E}_0)/\ln \beta} = \left(\frac{\mathcal{E}}{\mathcal{E}_0}\right)^{\ln(1-P)/\ln \beta} \quad (a^{\ln x} = x^{\ln a})$$

Key result

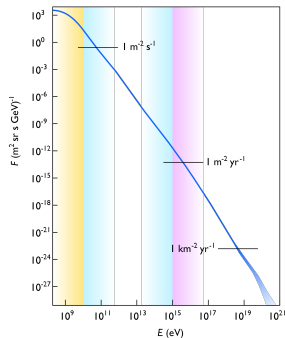
$$N(\mathcal{E}) d\mathcal{E} \propto \mathcal{E}^{-s} d\mathcal{E}, \quad s = 1 - \frac{\ln(1 - P)}{\ln \beta} \simeq 1 + \frac{P}{\xi}$$

Only escape/gain matters. Strong shocks: $s = 2$ (ch. 14); cosmic rays: $\mathcal{E}^{-2.7}$.

Power laws: computed and observed



Monte Carlo of the cycle model (points) and $s = 1 - \ln(1 - P)/\ln \beta$ (lines). $P/\xi = 1.5$ gives $s \approx 2.7$.



S. Lafebre, CC BY-SA 3.0, after Swordy and De Angelis

Cosmic rays: $\propto E^{-2.7}$, steepening at the *knee* (3×10^{15} eV).

Summary

Mirror force	$F_{\parallel} = -\mu \partial B / \partial s$
First invariant	$\mu = \frac{1}{2} m v_{\perp}^2 / B, \Phi = \pi r_L^2 B$
Loss cone	$\sin^2 \alpha_{lc} = B_{min} / B_{max}$
Oscillator	E/ω invariant
Second, third	$J = \oint m v_{\parallel} ds$, drift-shell Φ
Dipole	$\sin^2 \alpha_{lc} = L^{-5/2} (4L - 3)^{-1/2}$
Fermi	$\Delta \mathcal{E} / \mathcal{E} \sim (V/v)^2$ or V/v
Spectrum	$s \simeq 1 + P/\xi$

- ▶ slow changes conserve actions
- ▶ strong field reflects: mirrors, belts
- ▶ moving mirrors accelerate: cosmic rays

Next: module 5a, kinetic theory: from Klimontovich to Vlasov; then Landau damping.

Problems: set 2.