

Plasma Physics and Dynamo Theory • Module 4a

Magnetic mirrors and the first invariant

Alfio Bonanno

INAF – Osservatorio Astrofisico di Catania

Outline

1. The first invariant and magnetic mirrors
2. The loss cone

Reading

Chen, ch. 2

Fitzpatrick, ch. 2

Northrop (1963)

Notes: chapter 4, §4.1–4.2

Problems: set 2

Motion along the field

Chapter 3: the guiding centre drifts *across* $\mathbf{B}$. Now: motion *along* $\mathbf{B}$, and the quantities conserved while the fields change slowly.

Questions

- ▶ why do strong-field regions reflect particles?
- ▶ how does the Earth trap the radiation belts?
- ▶ how do moving magnetic fields accelerate cosmic rays?

Answer: adiabatic invariants

gyration $\mu = \frac{1}{2}mv_{\perp}^2/B$

bounce $J = \oint mv_{\parallel} ds$

drift flux Φ of the drift shell

The guiding-centre picture: separation of scales

fast: gyration

frequency Ω_c , radius r_L , phase φ

slow: guiding centre along B

speed $v_{\parallel}$, field varies on a length L

The guiding-centre description holds when

$$\epsilon_1 = \frac{r_L}{L} \ll 1, \quad \epsilon_2 = \frac{v_{\parallel}}{L \Omega_c} \ll 1$$

Averaging over φ , $(\mathbf{v}_{\perp}, \varphi)$ are replaced by μ :

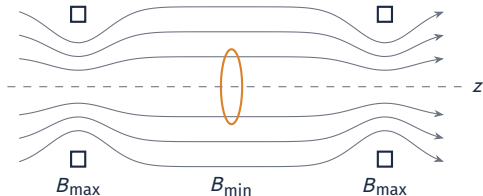
Key result: guiding-centre equations along B

$$\frac{ds}{dt} = v_{\parallel}, \quad m \frac{dv_{\parallel}}{dt} = -\mu \frac{\partial B}{\partial s}, \quad \mu = \text{const}$$

1 MeV proton at $4R_E$, $B = B_E(R_E/r)^3 \approx 5$ mG: $r_L \approx 300$ km $\ll L_B = r/3 \approx 8500$ km;
 $\Omega_c \approx 46 \gg \omega_b \approx 0.7$ s $^{-1}$.

The first invariant and magnetic mirrors

An axisymmetric mirror field



$\mathbf{B} = (B_r, 0, B_z)$, $\partial_\theta = 0$. Converging lines need $B_r \neq 0$:

$$\nabla \cdot \mathbf{B} = \frac{1}{r} \frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0$$

Near the axis $\partial_z B_z \simeq (\partial_z B_z)_0$, independent of r ; integrate from $r = 0$:

$$rB_r = - \int_0^r r' \frac{\partial B_z}{\partial z} dr' \simeq - \frac{r^2}{2} \left(\frac{\partial B_z}{\partial z} \right)_0 \Rightarrow B_r \simeq - \frac{r}{2} \left(\frac{\partial B_z}{\partial z} \right)_0$$

The Lorentz force in the mirror field

$\mathbf{F} = (q/c) \mathbf{v} \times \mathbf{B}$ with $B_\theta = 0$:

$$F_r = \frac{q}{c} \underbrace{v_\theta B_z}_1, \quad F_\theta = \frac{q}{c} \left(\underbrace{-v_r B_z}_2 + \underbrace{v_z B_r}_3 \right), \quad F_z = \frac{q}{c} \underbrace{(-v_\theta B_r)}_4$$

- ▶ 1, 2: Larmor gyration (chapter 3)
- ▶ 3: v_z with $B_r \neq 0$ drives motion around the axis: the grad- B drift in θ
- ▶ 4: **new**: a force *along* the axis

With $B_r \simeq -\frac{r}{2}(\partial_z B_z)_0$:

$$F_z = -\frac{q}{c} v_\theta B_r = \frac{q}{2c} v_\theta r \left(\frac{\partial B_z}{\partial z} \right)_0$$

The mirror force

Guiding centre on the axis: $r = r_L$, $v_\theta = -\text{sign}(q) v_\perp$ (ions turn in the $-\theta$ sense), constant over the gyration:

$$F_z = -\frac{|q|}{2c} v_\perp r_L \frac{\partial B_z}{\partial z} = -\frac{|q|}{2c} v_\perp \frac{mc v_\perp}{|q| B} \frac{\partial B_z}{\partial z} = -\frac{\frac{1}{2} m v_\perp^2}{B} \frac{\partial B_z}{\partial z}$$

Key result: mirror force

$$F_{\parallel} = -\mu \frac{\partial B}{\partial s}, \quad \mu = \frac{\frac{1}{2} m v_\perp^2}{B}$$

- ▶ the parallel part of $-\mu \nabla B$ found in chapter 3: a diamagnetic dipole
- ▶ same μ as the current loop: $\mu = IA/c$, $I = |q|\omega_c/2\pi$, $A = \pi r_L^2$

μ is invariant (static field)

Multiply $m dv_{\parallel}/dt = -\mu \partial B/\partial s$ by $v_{\parallel} = ds/dt$:

$$\frac{d}{dt} \left(\frac{1}{2} m v_{\parallel}^2 \right) = -\mu \frac{\partial B}{\partial s} \frac{ds}{dt} = -\mu \frac{dB}{dt}$$

B does no work: energy conserved, with $\frac{1}{2} m v_{\perp}^2 = \mu B$,

$$\frac{d}{dt} \left(\frac{1}{2} m v_{\parallel}^2 + \mu B \right) = 0$$

Substitute the first result:

$$-\mu \frac{dB}{dt} + \mu \frac{dB}{dt} + B \frac{d\mu}{dt} = B \frac{d\mu}{dt} = 0$$

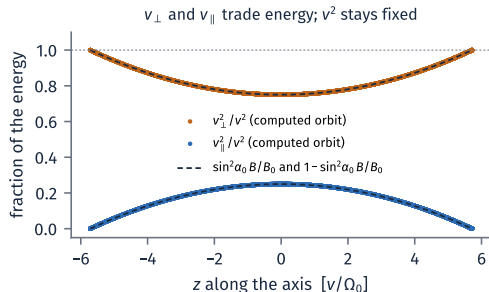
Key result

$\mu = \text{const}$: as B increases, $v_{\perp}$ grows and r_L shrinks, but μ stays fixed.

Adiabatic coupling of $v_{\perp}$ and $v_{\parallel}$

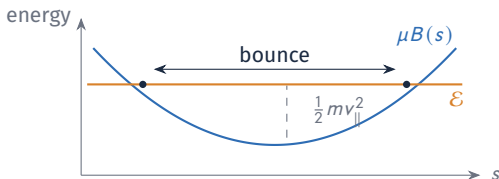
Pitch angle α : angle between $\mathbf{v}$ and $\mathbf{B}$, $\sin \alpha = v_{\perp}/v$, $\cos \alpha = v_{\parallel}/v$ (α_0 where $B = B_0$). With μ and $\mathcal{E}$ fixed:

$$\frac{1}{2}mv_{\perp}^2 = \mu B(s), \quad \frac{1}{2}mv_{\parallel}^2 = \mathcal{E} - \mu B(s) \quad \Rightarrow \quad \frac{v_{\perp}^2(s)}{v^2} = \sin^2 \alpha_0 \frac{B(s)}{B_0}, \quad \frac{\sin^2 \alpha}{B} = \text{const}$$



- ▶ $v_{\perp}$ slaved to B : energy flows $v_{\parallel} \rightarrow v_{\perp}$ as B grows, and back
- ▶ agent: $-(q/c) v_{\theta} B_r$; no work, only rotation of $\mathbf{v}$
- ▶ **adiabatic**: slow ($\epsilon_{1,2} \ll 1$), reversible

The magnetic mirror



Energy conservation with μ fixed:

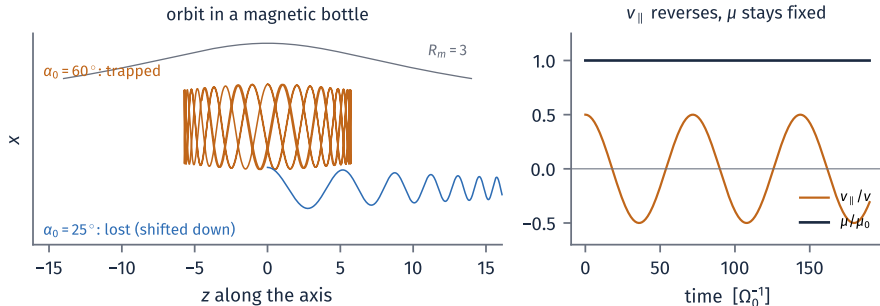
$$\frac{1}{2}mv_{\parallel}^2 = \mathcal{E} - \mu B(s)$$

- ▶ $\mu B(s)$ acts as a **potential** for the parallel motion
- ▶ where $\mu B = \mathcal{E}$: $v_{\parallel} = 0$, the particle turns back

Key result

Between two strong-field regions the particle bounces back and forth: it is **trapped** in a magnetic bottle.

A trapped orbit, computed



Full Lorentz force, $R_m = 3$, no guiding-centre approximation.

- ▶ $\alpha_0 = 60^\circ$: reflected where $B = B_0/\sin^2 \alpha_0 = 1.33 B_0$; μ constant to $< 10^{-3}$
- ▶ $\alpha_0 = 25^\circ$: $\sin^2 \alpha_0 = 0.18 < 1/R_m$, in the loss cone: escapes

μ is invariant (time-varying field)

B changes slowly at fixed position: $\nabla \times \mathbf{E} = -c^{-1} \partial_t \mathbf{B}$ does work. Per gyration:

$$\delta\left(\frac{1}{2}mv_{\perp}^2\right) = \oint q\mathbf{E} \cdot d\mathbf{l} = q \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{q}{c} \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

Diamagnetic: $\mathbf{B} \cdot d\mathbf{S} < 0$ (ions), > 0 (electrons); with $\delta B = \dot{B} 2\pi/\omega_c$:

$$\delta\left(\frac{1}{2}mv_{\perp}^2\right) = \frac{|q|}{c} \pi r_L^2 \dot{B} = \frac{|q|\omega_c r_L^2}{2c} \delta B = \frac{\frac{1}{2}mv_{\perp}^2}{B} \delta B = \mu \delta B$$

But $\frac{1}{2}mv_{\perp}^2 = \mu B \Rightarrow \delta\left(\frac{1}{2}mv_{\perp}^2\right) = \mu \delta B + B \delta\mu$:

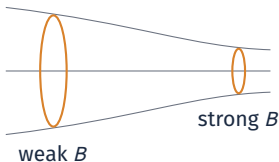
Key result

$\delta\mu = 0$ also in slowly time-varying fields. $T_{\perp} \propto B$: magnetic compression heats (**betatron heating**).

Flux through the orbit

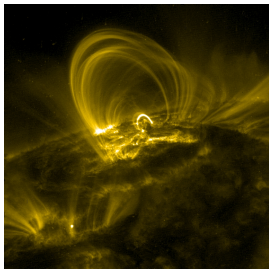
Equivalent statement:

$$\Phi = \pi r_L^2 B = \pi \frac{v_{\perp}^2}{\omega_c^2} B = \pi \frac{m^2 c^2 v_{\perp}^2}{q^2 B} = \frac{2\pi m c^2}{q^2} \mu = \text{const}$$



The orbit shrinks as B grows, always enclosing the same flux: the particle **follows the field lines**. This is the single-particle version of flux freezing (ch. 12).

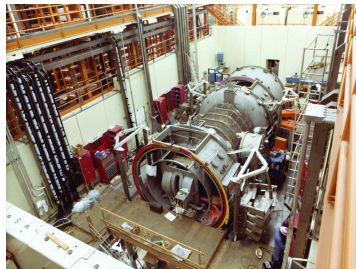
Magnetic bottles in nature and in the laboratory



NASA/TRACE, 171 Å

Coronal loops: B strong at the footpoints, weak at the top. Flare electrons bounce; those in the loss cone hit the chromosphere and shine in hard X-rays.

M4a • Magnetic mirrors



LLNL / R. H. Hirschfeld

Tandem Mirror Experiment (Livermore, 1979). End losses through the loss cone, refilled by collisions, made mirrors lose out to tokamaks.

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The loss cone

The loss cone: derivation

At the centre, $B = B_0 = B_{\min}$: speed v_0 , components $v_{\perp 0}$, $v_{\parallel 0}$. Recall the pitch angle, $\sin \alpha = v_{\perp}/v$. At the turning point all energy is perpendicular, $v_{\perp} = v_0$. Conservation of μ :

$$\frac{v_{\perp 0}^2}{B_0} = \frac{v_0^2}{B_{\text{turn}}} \Rightarrow B_{\text{turn}} = \frac{B_0}{\sin^2 \alpha_0}$$

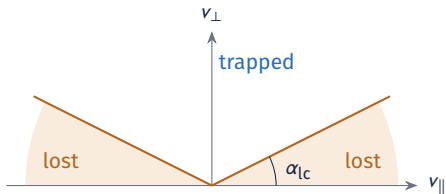
Reflected if $B_{\text{turn}} < B_{\max}$; escapes if $\sin^2 \alpha_0 < B_0/B_{\max}$:

Key result: loss cone

$$\sin^2 \alpha_{\text{lc}} = \frac{B_{\min}}{B_{\max}} = \frac{1}{R_m} \quad (R_m = \text{mirror ratio})$$

Independent of q and m . Particles with $v_{\perp} = 0$ ($\mu = 0$) always escape.

The loss cone in velocity space



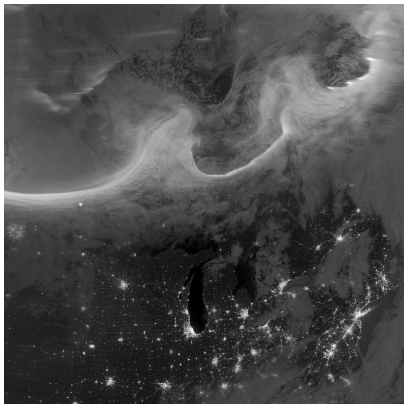
Isotropic plasma: fraction in the two cones = solid-angle fraction,

$$f_{\text{lost}} = \frac{2 \cdot 2\pi(1 - \cos \alpha_{lc})}{4\pi} = 1 - \sqrt{1 - R_m^{-1}}$$

$R_m = 5$: $\alpha_{lc} = 26.6^\circ$, 11% lost at once.

- ▶ collisions scatter in pitch angle and **refill** the loss cone
- ▶ electrons collide faster, escape first: an ambipolar potential holds them back
- ▶ end losses doomed mirror machines; empty loss cones drive instabilities

Where lost particles go: the aurora



NASA Earth Observatory / Suomi NPP VIIRS

- ▶ the Earth's field is a giant bottle: particles mirror near the poles
- ▶ waves and collisions scatter them into the loss cone; they reach the atmosphere at the feet of the field lines
- ▶ the band of light over Canada (night-time visible image from orbit) traces the field lines from $L \approx 5-10$: the **auroral oval**
- ▶ cities below, for scale: Great Lakes, Chicago, the East Coast

So far, and next

In this module

- ▶ guiding centre: $r_L/L \ll 1$, $v_{\parallel}/L\Omega_c \ll 1$; average over the gyration
- ▶ mirror force $F_{\parallel} = -\mu \partial B / \partial s$; μ conserved
- ▶ adiabatic exchange: $v_{\perp}^2 \propto B$, $\frac{1}{2}mv_{\parallel}^2 = \mathcal{E} - \mu B$
- ▶ loss cone $\sin^2 \alpha_{lc} = B_{\min}/B_{\max}$
- ▶ fails if $r_L \sim L$ or fields vary at Ω_c : μ broken, loss cone refilled

Next: module 4b: invariants as actions, the Van Allen belts and Fermi acceleration.