Plasma Physics and Dynamo Theory • Module 3b

Orbits: inhomogeneous fields and fluid currents

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Outline

1. Inhomogeneous fields
2. From particles to fluid
3. Validity of the guiding-centre picture

Reading

Chen, ch. 2
Sturrock, ch. 3
Goldston & Rutherford, ch. 2–3
Northrop (1963)

Notes: chapter 3, §3.3–3.5
Problems: set 2

Where we are

Key result: recap

- gyration at ω_c on a circle r_L around the guiding centre
- magnetic moment $\mu = \frac{1}{2} m v_\perp^2 / B$, antiparallel to $\boldsymbol{B}$
- any force $\boldsymbol{F}$: drift $(c/q)\boldsymbol{F} \times \boldsymbol{B}/B^2$; $\boldsymbol{E} \times \boldsymbol{B}$ for all species
- today: fields that vary in space, and how drifts add up to fluid currents

Inhomogeneous fields

The grad-B drift (1): orbit average

$\boldsymbol{B} = B_z(x)\,\hat{\boldsymbol{z}}$, with $r_L|\nabla B| \ll B$. Expand about the guiding centre x_0:

$$F_x = \frac{q}{c}\,v_y B_z(x) = \frac{q}{c}\,v_y B_0 + \frac{q}{c}\,v_y (x - x_0)\,\frac{\partial B}{\partial x}$$

Average over the unperturbed orbit: $v_x = v_\perp \cos\Omega t$, $v_y = -v_\perp \sin\Omega t$, $x - x_0 = (v_\perp/\Omega)\sin\Omega t$:

$$\langle v_y \rangle = 0, \qquad \langle v_y (x - x_0)\rangle = -\frac{v_\perp^2}{\Omega}\langle \sin^2 \Omega t\rangle = -\frac{v_\perp^2}{2\Omega} \;\Rightarrow\; \langle F_x\rangle = -\frac{q}{c}\frac{v_\perp^2}{2\Omega}\frac{\partial B}{\partial x} = -\frac{\frac{1}{2}mv_\perp^2}{B}\frac{\partial B}{\partial x}$$

Key result

$$\langle \boldsymbol{F}\rangle = -\mu\,\nabla B \qquad (\langle F_y\rangle = 0;\ \text{a diamagnetic dipole, pushed to weak field})$$

The grad-B drift (2)

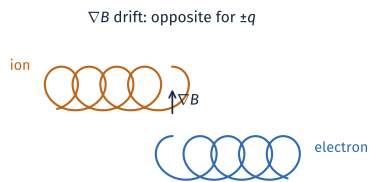
Insert $\langle \boldsymbol{F} \rangle = -\mu \nabla B$ in the force drift:

$$\boldsymbol{v}_{\nabla B} = \frac{c}{q} \frac{(-\mu \nabla B) \times \boldsymbol{B}}{B^2} = \frac{\mu c}{q} \frac{\boldsymbol{B} \times \nabla B}{B^2}$$

Key result: grad-B drift

$$\boldsymbol{v}_{\nabla B} = \frac{v_\perp^2}{2\Omega} \frac{\hat{\boldsymbol{b}} \times \nabla B}{B}, \qquad v_{\nabla B} \sim \frac{v_\perp}{2} \frac{r_L}{L_B}$$

- smaller than $v_\perp$ by r_L / L_B
- opposite for $\pm q$: current
- radius smaller where B is larger: orbit does not close



The curvature drift

Particle sliding along a curved line at $v_\parallel$: centrifugal force in its frame,

$$\boldsymbol{F}_{\mathrm{cf}} = \frac{m v_\parallel^2}{R_c}\,\hat{\boldsymbol{e}}_r = m v_\parallel^2\,\frac{\boldsymbol{R}_c}{R_c^2}$$

$\boldsymbol{R}_c$: from the centre of curvature to the line.

Key result: curvature drift

$$\boldsymbol{v}_R = \frac{c}{q}\,\frac{\boldsymbol{F}_{\mathrm{cf}}\times\boldsymbol{B}}{B^2} = \frac{m c v_\parallel^2}{q}\,\frac{\boldsymbol{R}_c\times\boldsymbol{B}}{R_c^2 B^2} = \frac{v_\parallel^2}{\Omega}\,\hat{\boldsymbol{b}}\times\boldsymbol{\kappa}, \qquad \boldsymbol{\kappa} = (\hat{\boldsymbol{b}}\cdot\nabla)\hat{\boldsymbol{b}} = -\frac{\boldsymbol{R}_c}{R_c^2}$$

A curved field always has a gradient too. Next: both together in a vacuum field.

Curvature + grad-B in a vacuum field

Vacuum or low pressure: $\nabla \times \boldsymbol{B} = 0$. Local cylindrical coordinates, $\boldsymbol{B} = B_\theta(r)\hat{\boldsymbol{\theta}}$:

$$(\nabla \times \boldsymbol{B})_z = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rB_\theta) = 0 \;\Rightarrow\; B_\theta \propto \frac{1}{r} \;\Rightarrow\; \frac{\nabla B}{B} = -\frac{\hat{\boldsymbol{e}}_r}{r} = -\frac{\boldsymbol{R}_c}{R_c^2}$$

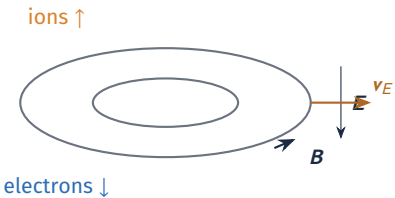
Then $\boldsymbol{v}_{\nabla B} = \dfrac{mcv_\perp^2}{2qB}\dfrac{\boldsymbol{B}\times\nabla B}{B^2} = \dfrac{mcv_\perp^2}{2q}\dfrac{\boldsymbol{R}_c\times\boldsymbol{B}}{R_c^2B^2}$: same direction as $\boldsymbol{v}_R$.

Key result

$$\boldsymbol{v}_R + \boldsymbol{v}_{\nabla B} = \frac{mc}{q}\frac{\boldsymbol{R}_c\times\boldsymbol{B}}{R_c^2B^2}\left(v_\parallel^2 + \tfrac{1}{2}v_\perp^2\right)$$

Maxwellian: $\langle v_\parallel^2 + \frac{1}{2}v_\perp^2\rangle = T/m + T/m = 2T/m$, so $\langle v_R + v_{\nabla B}\rangle = \dfrac{2cT}{qBR_c} \propto \dfrac{T}{BR_c}$, independent of mass.

Why a simple torus cannot confine a plasma



Field lines are circles around the axis ($B \propto 1/R$):

1. curvature + grad-B: ions up, electrons down
2. charge separation $\Rightarrow$ vertical $\boldsymbol{E}$
3. $\boldsymbol{v}_E = c\boldsymbol{E} \times \boldsymbol{B}/B^2$ outwards, for both species
4. the plasma hits the wall

Estimate

$T = 1$ keV, $B = 10^4$ G, $R_c = 1$ m: $v = 2cT/eBR_c \approx 2$ km s^{-1}: the wall is reached in ~ 1 ms.

Cure: twist the field lines into helices, so particles spend equal times above and below the midplane. Tokamak (plasma current), stellarator (external coils).

From particles to fluid

A paradox

Uniform $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, density gradient along x, uniform T.

Particle view

uniform B: no grad-B, no curvature drift.
No guiding centre moves.

Fluid view

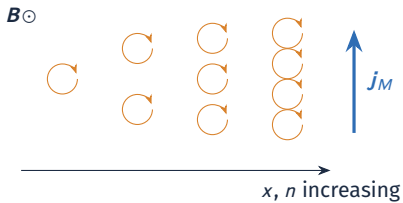
force balance $\boldsymbol{j} \times \boldsymbol{B}/c = \nabla p$ requires

$$\boldsymbol{j} = c\,\frac{\hat{\boldsymbol{b}} \times \nabla p}{B} \neq 0$$

Key result

Where does the current come from? From the **gyration** itself: each particle is a current loop, and loops do not cancel where their density varies.

The magnetisation current



Across a vertical line: loops on the left contribute their (downward) right sides, loops on the right their (upward) left sides. More loops on the right: net upward current, along $\hat{\boldsymbol{b}} \times \nabla n$.

Magnetisation = moment per unit volume, with $p_\perp = \sum n \langle \frac{1}{2} m v_\perp^2 \rangle$:

$$\boldsymbol{M} = \sum_s n_s \langle \boldsymbol{\mu}_s \rangle = -\sum_s n_s \left\langle \frac{\frac{1}{2} m_s v_\perp^2}{B} \right\rangle \hat{\boldsymbol{b}} = -\frac{p_\perp}{B}\, \hat{\boldsymbol{b}}, \qquad \boldsymbol{j}_M = c\, \nabla \times \boldsymbol{M}$$

Summing all contributions

Straight lines, $\boldsymbol{B} = B(x)\hat{\boldsymbol{z}}$, isotropic $p(x)$. **Drift current** (grad-B, $\boldsymbol{v}_{\nabla B} = (\mu c/q)\,\hat{\boldsymbol{b}} \times \nabla B/B$):

$$\boldsymbol{j}_{\nabla B} = \sum n q\, \boldsymbol{v}_{\nabla B} = c\Big(\sum n\mu\Big)\frac{\hat{\boldsymbol{b}} \times \nabla B}{B} = c\, p\, \frac{\hat{\boldsymbol{b}} \times \nabla B}{B^2}$$

Magnetisation current, $\boldsymbol{M} = -(p/B)\hat{\boldsymbol{z}}$, $\nabla \times (f\,\hat{\boldsymbol{z}}) = \nabla f \times \hat{\boldsymbol{z}}$:

$$\boldsymbol{j}_M = -c\,\nabla\Big(\frac{p}{B}\Big) \times \hat{\boldsymbol{z}} = c\,\frac{\hat{\boldsymbol{b}} \times \nabla p}{B} - c\, p\, \frac{\hat{\boldsymbol{b}} \times \nabla B}{B^2}$$

Key result: diamagnetic current

$$\boldsymbol{j}_\perp = \boldsymbol{j}_{\nabla B} + \boldsymbol{j}_M = c\,\frac{\hat{\boldsymbol{b}} \times \nabla p}{B}, \qquad \frac{\boldsymbol{j}_\perp \times \boldsymbol{B}}{c} = (\hat{\boldsymbol{b}} \times \nabla p) \times \hat{\boldsymbol{b}} = \nabla_\perp p$$

Particles and fluid agree

- the grad-B terms cancel exactly: the total current depends only on ∇p
- particle and fluid pictures agree only when the magnetisation current is included
- the fluid velocity is *not* the guiding-centre velocity

Key result

Two-fluid language: each species carries the **diamagnetic drift**

$$\boldsymbol{u}_{*s} = c\,\frac{\hat{\boldsymbol{b}} \times \nabla p_s}{q_s n_s B}$$

a fluid velocity that no individual guiding centre has (ch. 9).

Curved fields add $c(p_\parallel - p_\perp)\,\hat{\boldsymbol{b}} \times \boldsymbol{\kappa}/B$, zero for isotropic p.

Validity of the guiding-centre picture

Ordering and summary of drifts

Small parameters: $\epsilon = r_L/L \ll 1$, $\omega/\omega_c \ll 1$. Drifts are first order: $v_{\nabla B}/v_\perp \sim v_R/v_\parallel \sim \epsilon$, $v_P/v_E \sim \omega/\omega_c$.

drift	velocity	depends on q?	current?
$\boldsymbol{E}\times\boldsymbol{B}$	$c\,\boldsymbol{E}\times\boldsymbol{B}/B^2$	no	no
force $\boldsymbol{F}$	$(c/q)\,\boldsymbol{F}\times\boldsymbol{B}/B^2$	yes	yes
gravitational	$(mc/q)\,\boldsymbol{g}\times\boldsymbol{B}/B^2$	yes, $\propto m$	yes (ions)
grad-B	$(v_\perp^2/2\Omega)\,\hat{\boldsymbol{b}}\times\nabla B/B$	yes	yes
curvature	$(v_\parallel^2/\Omega)\,\hat{\boldsymbol{b}}\times\boldsymbol{\kappa}$	yes	yes
polarisation	$(1/\Omega)(c/B)\,\partial_t\boldsymbol{E}_\perp$	yes, $\propto m$	yes (ions)

Summary

Guiding-centre velocity to first order:

$$\boldsymbol{v}_{\mathrm{gc}} = v_{\parallel}\hat{\boldsymbol{b}} + \boldsymbol{v}_E + \frac{\hat{\boldsymbol{b}}}{\Omega} \times \left(\frac{v_{\perp}^2}{2} \frac{\nabla B}{B} + v_{\parallel}^2 \boldsymbol{\kappa} + \frac{\mathrm{d}\boldsymbol{v}_E}{\mathrm{d}t} \right)$$

- gyration: $\omega_c = |q|B/mc$, $r_L = v_{\perp}/\omega_c$, $\mu = \frac{1}{2} m v_{\perp}^2 / B$
- $\boldsymbol{E} \times \boldsymbol{B}$: no current; all other drifts carry currents
- drift + magnetisation currents $= c\,\hat{\boldsymbol{b}} \times \nabla p / B$

Still missing: the motion *along* $\boldsymbol{B}$, governed by the mirror force $-\mu\,\partial B/\partial s$ and the conservation of μ.

Next: module 4a, magnetic mirrors; then radiation belts and Fermi acceleration.

Problems: set 2.