

Orbits: gyration and drifts

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Outline

1. Gyromotion
2. $\mathbf{E} \times \mathbf{B}$ and force drifts
3. Time-varying fields: the polarisation drift

Reading

Chen, ch. 2

Sturrock, ch. 3

Goldston & Rutherford, ch. 2–3

Notes: chapter 3, §3.1–3.2, 3.3.4

Problems: set 2

Single particles in given fields

Session 1: the plasma as a statistical system. Today: **one charge** in prescribed $\mathbf{E}$ and $\mathbf{B}$, ignoring the fields it creates.

Why bother?

- ▶ magnetospheres, radiation belts, cosmic rays: particles really are nearly independent
- ▶ magnetic confinement in the lab
- ▶ summed over particles, drifts and gyrations give the **fluid currents** of MHD

The key idea

fields vary slowly on the scale of one gyration:

$$\frac{r_L}{L} \ll 1, \quad \frac{\omega}{\omega_c} \ll 1$$

fast gyration around a **guiding centre** + slow motion of the guiding centre.

Gyromotion

Equation of motion

$$m \frac{d\mathbf{v}}{dt} = q \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right)$$

Multiply by $\mathbf{v}$:

$$\frac{d}{dt} \left(\frac{1}{2} m v^2 \right) = q \mathbf{v} \cdot \mathbf{E} + \frac{q}{c} \underbrace{\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B})}_{=0} = q \mathbf{v} \cdot \mathbf{E}$$

Key result

A static magnetic field does **no work**: it turns the velocity but does not change the energy. Only $\mathbf{E}$ changes the energy.

Uniform magnetic field

$\mathbf{E} = 0$, $\mathbf{B} = B \hat{\mathbf{z}}$: $\mathbf{v} \times \mathbf{B} = B(v_y, -v_x, 0)$. Components:

$$\dot{v}_x = \Omega v_y, \quad \dot{v}_y = -\Omega v_x, \quad \dot{v}_z = 0, \quad \Omega \equiv \frac{qB}{mc} \text{ (signed)}$$

► $v_{\parallel} = v_z = \text{const}$: no force along $\mathbf{B}$

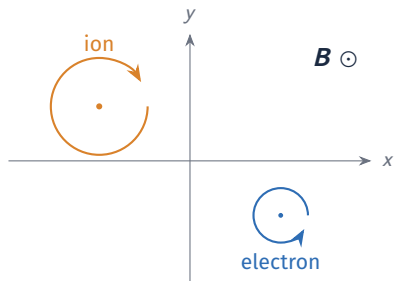
► $\ddot{v}_x = \Omega \dot{v}_y = -\Omega^2 v_x$: harmonic, at the cyclotron frequency $\omega_c = |\Omega| = \frac{|q|B}{mc}$

Complex form, $w = v_x + i v_y$: $\dot{w} = \Omega v_y - i \Omega v_x = -i \Omega w$, so

$$v_x + i v_y = v_{\perp} e^{-i(\Omega t + \varphi_0)}, \quad x + i y = X + i Y + \frac{i v_{\perp}}{\Omega} e^{-i(\Omega t + \varphi_0)}$$

circle around the **guiding centre** (X, Y) ; helix along $\mathbf{B}$.

Larmor radius and sense of rotation



Key result: Larmor radius

$$r_L = \frac{v_{\perp}}{\omega_c} = \frac{mcv_{\perp}}{|q|B}$$

- ▶ ions ($\Omega > 0$) rotate clockwise seen with B towards us; electrons anticlockwise
- ▶ $r_{Le}/r_{Li} = (m_e/m_i)^{1/2}$ at equal T
- ▶ relativistic: $m \rightarrow \gamma m$

Practical (B in G, T in eV, $v_{\perp} = (T/m)^{1/2}$): $\omega_{ce} = 1.76 \times 10^7 B \text{ s}^{-1}$, $r_{Le} = 2.38 T^{1/2}/B \text{ cm}$, $r_{Li} = 102 (AT)^{1/2}/ZB \text{ cm}$.

Estimate: gyration across the Universe

	B [G]	T [eV]	f_{ce}	r_{Le}	r_{Lp}
Solar corona	10	86	28 MHz	2 cm	1 m
Earth, $L = 6.6$	10^{-3}	10^3	3 kHz	0.8 km	30 km
Warm ionised ISM	5×10^{-6}	1	14 Hz	5 km	200 km
Tokamak core	5×10^4	10^4	140 GHz	$50 \mu\text{m}$	2 mm

Key result

Gyroradii are tiny compared with system sizes, gyrofrequencies are huge compared with the rates of change of the fields: the **separation of scales** behind the guiding-centre picture.

Magnetic moment and diamagnetism

A gyrating charge is a current loop: $I = |q|\omega_c/2\pi$, area $A = \pi r_L^2$.

$$\mu = \frac{IA}{c} = \frac{|q|\omega_c}{2\pi c} \pi r_L^2 = \frac{|q|\omega_c}{2c} \frac{v_\perp^2}{\omega_c^2} = \frac{|q|v_\perp^2}{2c} \frac{mc}{|q|B}$$

Key result: magnetic moment

$$\mu = \frac{\frac{1}{2}mv_\perp^2}{B} \quad (\text{perpendicular energy / field})$$

- ▶ for *both* signs of q the loop's own field opposes $\mathbf{B}$ inside the orbit: $\boldsymbol{\mu} = -\mu \hat{\mathbf{b}}$
- ▶ a plasma is **diamagnetic**: its particles weaken the field they move in
- ▶ macroscopically: $\rho + B^2/8\pi \simeq \text{const}$ across a magnetised plasma

$E \times B$ and force drifts

The $\mathbf{E} \times \mathbf{B}$ drift: derivation

Along $\mathbf{B}$: $m\dot{\mathbf{v}}_{\parallel} = qE_{\parallel}$ (in a plasma $E_{\parallel} \simeq 0$). Perpendicular:
 $m d\mathbf{v}_{\perp}/dt = q(\mathbf{E}_{\perp} + \mathbf{v}_{\perp} \times \mathbf{B}/c)$. Try $\mathbf{v}_{\perp} = \mathbf{v}_d + \mathbf{v}_g(t)$, $\mathbf{v}_d$ constant:

$$m \frac{d\mathbf{v}_g}{dt} = q \underbrace{\left(\mathbf{E}_{\perp} + \frac{1}{c} \mathbf{v}_d \times \mathbf{B} \right)}_{\text{constant: set to 0}} + \frac{q}{c} \mathbf{v}_g \times \mathbf{B}$$

Solve $\mathbf{E}_{\perp} + \mathbf{v}_d \times \mathbf{B}/c = 0$: cross with $\mathbf{B}$, use $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$:

$$\mathbf{B} \times \mathbf{E}_{\perp} + \frac{1}{c} \left[\mathbf{v}_d B^2 - \underbrace{\mathbf{B}(\mathbf{B} \cdot \mathbf{v}_d)}_0 \right] = 0 \quad \Rightarrow \quad \mathbf{v}_d = -c \frac{\mathbf{B} \times \mathbf{E}}{B^2}$$

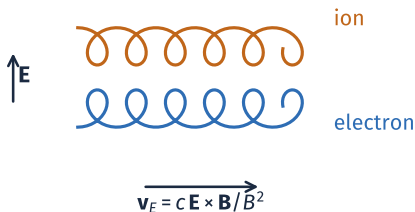
The $E \times B$ drift

Key result: $E \times B$ drift

$$\mathbf{v}_E = c \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

- ▶ independent of q , m , energy
- ▶ ions and electrons drift together: **no current**
- ▶ in the frame moving with $\mathbf{v}_E$, $\mathbf{E} = 0$
- ▶ needs $E_{\perp} < B$ ($v_E < c$)

$\mathbf{E} \times \mathbf{B}$ drift ($\mathbf{B}$ out of page)



Picture: moving along $\mathbf{E}$ the ion speeds up and its radius grows; moving against $\mathbf{E}$ it shrinks. The orbit does not close. The electron is pushed the other way but gyrates the other way: **same step**.

General force drift; gravity

Only qE entered: any force F constant over a gyration gives ($qE \rightarrow F$)

Key result: force drift

$$\mathbf{v}_F = \frac{c}{q} \frac{\mathbf{F} \times \mathbf{B}}{B^2} \quad \text{depends on the sign of } q \Rightarrow \text{current}$$

Gravity: $\mathbf{v}_g = \frac{mc}{q} \frac{\mathbf{g} \times \mathbf{B}}{B^2}$, ions faster, opposite to electrons:

$$\mathbf{j} = \sum_s n_s q_s \mathbf{v}_{g,s} = (n_i m_i + n_e m_e) c \frac{\mathbf{g} \times \mathbf{B}}{B^2} \simeq \rho c \frac{\mathbf{g} \times \mathbf{B}}{B^2}$$

Lorentz force on this current, with $(\mathbf{g} \times \mathbf{B}) \times \mathbf{B} = -B^2 \mathbf{g}_\perp$:

$$\frac{\mathbf{j} \times \mathbf{B}}{c} = \rho \frac{(\mathbf{g} \times \mathbf{B}) \times \mathbf{B}}{B^2} = -\rho \mathbf{g}_\perp \quad \text{the field holds the plasma up (prominences)}$$

Time-varying fields: the polarisation drift

The polarisation drift

$\mathbf{E}$ slowly varying $\Rightarrow \mathbf{v}_E$ changes $\Rightarrow$ the guiding centre must be accelerated. In the drift frame: inertial force $-m \dot{\mathbf{v}}_E$, giving

$$\mathbf{v}_P = \frac{c}{q} \frac{(-m \dot{\mathbf{v}}_E) \times \mathbf{B}}{B^2} = -\frac{mc}{qB^2} \left(\frac{c \dot{\mathbf{E}} \times \mathbf{B}}{B^2} \right) \times \mathbf{B} = -\frac{mc^2}{qB^4} (\dot{\mathbf{E}} \times \mathbf{B}) \times \mathbf{B}$$

With $(\dot{\mathbf{E}} \times \mathbf{B}) \times \mathbf{B} = -B^2 \dot{\mathbf{E}}_{\perp}$:

Key result: polarisation drift

$$\mathbf{v}_P = \frac{mc^2}{qB^2} \frac{\partial \mathbf{E}_{\perp}}{\partial t}, \quad \frac{v_P}{v_E} \sim \frac{\omega}{\omega_c}$$

$\propto m/q$: ions dominate. Polarisation current $\mathbf{j}_P \simeq (\rho c^2/B^2) \partial_t \mathbf{E}_{\perp}$.

The magnetised plasma as a dielectric

Ampère, for fields slower than ω_{ci} :

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j}_P + \frac{1}{c} \frac{\partial \mathbf{E}_\perp}{\partial t} = \frac{1}{c} \left(1 + \frac{4\pi\rho c^2}{B^2} \right) \frac{\partial \mathbf{E}_\perp}{\partial t}$$

Key result

$$\epsilon_\perp = 1 + \frac{4\pi\rho c^2}{B^2} = 1 + \frac{c^2}{v_A^2}, \quad v_A = \frac{B}{\sqrt{4\pi\rho}} \text{ (Alfvén speed)}$$

- ▶ usually $v_A \ll c$: $\epsilon_\perp$ is enormous ($\sim 10^5$ in the corona)
- ▶ low-frequency EM waves travel at $c/\sqrt{\epsilon_\perp} \simeq v_A$: **Alfvén waves** (ch. 12), found here from single orbits

So far, and next

In this module

- ▶ gyration: $\omega_c = |q|B/mc$, $r_L = v_\perp/\omega_c$,
 $\mu = \frac{1}{2}mv_\perp^2/B$ (diamagnetic)
- ▶ $\mathbf{v}_E = c\mathbf{E} \times \mathbf{B}/B^2$: no current; force drift
 $(c/q)\mathbf{F} \times \mathbf{B}/B^2$
- ▶ polarisation drift $\propto m/q$; $\epsilon_\perp = 1 + c^2/v_A^2$

Next: module 3b: inhomogeneous fields (grad- B and curvature drifts) and the fluid current.