

Resistivity, runaway and bremsstrahlung

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Outline

1. Resistivity and runaway
2. Radiation from collisions

Reading

Spitzer (1962), ch. 5

Fitzpatrick, ch. 3

Rybicki & Lightman, ch. 5

Notes: chapter 2, §2.4–2.5

Problems: set 1

Where we are

Key result: recap

- ▶ collisions are a diffusion in velocity space, strength $\ln \Lambda$
- ▶ $\nu_{ei} = 2.91 \times 10^{-6} Z n_e \ln \Lambda T_{\text{ev}}^{-3/2} \text{ s}^{-1}$, and $\nu(\nu) \propto \nu^{-3}$
- ▶ energy exchange slower by $2m_e/m_i$
- ▶ today: what collisions do to currents, and how colliding plasmas radiate

Resistivity and runaway

Ohm's law from friction

Uniform $\mathbf{E}$; ions at rest; electrons drift with $\mathbf{u}_e$. Electron momentum equation with friction:

$$m_e n_e \frac{d\mathbf{u}_e}{dt} = -en_e \mathbf{E} - m_e n_e \nu_{ei} \mathbf{u}_e$$

Steady state after a few collision times: $\mathbf{u}_e = -e\mathbf{E}/(m_e \nu_{ei})$, and

$$\mathbf{j} = -en_e \mathbf{u}_e = \frac{n_e e^2}{m_e \nu_{ei}} \mathbf{E} \equiv \sigma \mathbf{E}$$

Key result: the density cancels, only T (and $\ln \Lambda$) matters

$$\eta_{\text{res}} = \frac{1}{\sigma} = \frac{m_e \nu_{ei}}{n_e e^2} = \frac{4\sqrt{2}\pi}{3} \frac{Z e^2 m_e^{1/2} \ln \Lambda}{T_e^{3/2}}$$

Spitzer resistivity

The current is carried mostly by the faster electrons, which collide less ($\nu \propto v^{-3}$). Kinetic theory (Spitzer & Härm 1953, ch. 7), along $\mathbf{B}$, $Z = 1$: smaller by 1.96.

Key result: Spitzer resistivity

$$\eta_{\parallel} \simeq 0.51 \frac{m_e v_{ei}}{n_e e^2} = 5.9 \times 10^{-15} \frac{Z \ln \Lambda}{T_{\text{eV}}^{3/2}} \text{ s}$$

SI: $5.2 \times 10^{-5} Z \ln \Lambda T_{\text{eV}}^{-3/2} \Omega \text{ m}$. In Gaussian units resistivity is a time.

- ▶ perpendicular to $\mathbf{B}$: the simple estimate, $\eta_{\perp} \simeq 2\eta_{\parallel}$
- ▶ a plasma at 10^7 K conducts about as well as copper; the 10^6 K corona has ~ 100 times the resistivity of copper

Magnetic diffusivity

Ohm ($\mathbf{E} = \eta_{\text{res}} \mathbf{j}$) + Ampère ($\mathbf{j} = \frac{c}{4\pi} \nabla \times \mathbf{B}$) + Faraday ($\partial_t \mathbf{B} = -c \nabla \times \mathbf{E}$), uniform η_{res} :

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \left(\eta_{\text{res}} \frac{c}{4\pi} \nabla \times \mathbf{B} \right) = -\frac{c^2 \eta_{\text{res}}}{4\pi} \nabla \times \nabla \times \mathbf{B} = \frac{c^2 \eta_{\text{res}}}{4\pi} \nabla^2 \mathbf{B}$$

using $\nabla \times \nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\nabla^2 \mathbf{B}$.

Key result: Spitzer magnetic diffusivity

$$\eta = \frac{c^2 \eta_{\text{res}}}{4\pi} \simeq 4.2 \times 10^5 \frac{Z \ln \Lambda}{T_{\text{eV}}^{3/2}} \text{ cm}^2 \text{ s}^{-1} \simeq 10^{13} \left(\frac{\ln \Lambda}{20} \right) T_{\text{K}}^{-3/2} \text{ cm}^2 \text{ s}^{-1}$$

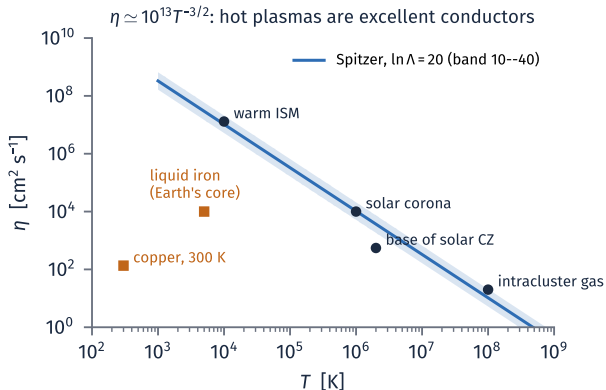
Ohmic decay of a structure of size L : $\tau_\eta \sim L^2/\eta$. Advection by a flow U against diffusion: $R_m = UL/\eta$.

Magnetic diffusivity across the Universe

	T [K]	η [cm ² /s]	L [cm]	$\tau_\eta = L^2/\eta$	$R_m = UL/\eta$
Solar corona (loop)	10^6	10^4	10^9	3×10^6 yr	10^{12}
Base of solar convection zone	2×10^6	5×10^2	2×10^{10}	3×10^{10} yr	10^{11}
Warm ionised ISM	10^4	10^7	3×10^{18}	3×10^{22} yr	10^{17}
Intracluster medium	10^8	20	3×10^{23}	10^{38} yr	10^{29}
Earth's outer core (liquid Fe)	5×10^3	10^4	3.5×10^8	4×10^5 yr	10^3

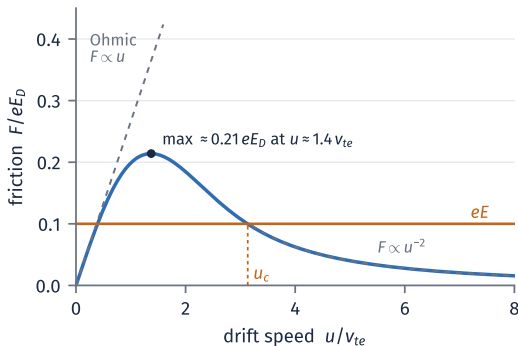
- ▶ **Earth:** free decay $L^2/\pi^2\eta \approx 4 \times 10^4$ yr $\ll$ 4.5 Gyr, yet a field for ≥ 3.5 Gyr $\Rightarrow$ it must be regenerated: **a dynamo**
- ▶ **plasmas:** Ohmic times far exceed the age of the Universe and R_m is enormous: field lines are frozen in (ch. 12)
- ▶ **Sun:** the field reverses every 11 yr, 10^9 times faster than Ohmic decay: the diffusion is turbulent (ch. 17)

Spitzer diffusivity versus temperature



- ▶ $\eta \propto T^{-3/2}$, almost independent of density
- ▶ above $\sim 10^7$ K a plasma conducts better than copper
- ▶ liquid iron is not a plasma: its η comes from metallic conduction and is large, which is why the Earth needs a dynamo

Friction on drifting electrons



Electrons drifting at u through ions (ch. 7):

$$F(u) = \frac{4\pi n_e e^4 \ln \Lambda}{m_e u^2} \psi\left(\frac{u}{\sqrt{2} v_{te}}\right)$$

$$\psi(x) = \operatorname{erf} x - \frac{2}{\sqrt{\pi}} x e^{-x^2}$$

$u \gg v_{te}$: $\psi \rightarrow 1$, $F \propto u^{-2}$ **decreases**.

$u \ll v_{te}$: $\psi \simeq \frac{4}{3\sqrt{\pi}} x^3$, so $F \simeq \frac{4\pi n_e e^4 \ln \Lambda}{m_e u^2} \cdot \frac{4}{3\sqrt{\pi}} \cdot \frac{u^3}{2\sqrt{2} v_{te}^3} = m_e v_{ei} u$: **exactly Ohm's law** with the Maxwellian-averaged v_{ei} .

The Dreicer field and runaway electrons

The friction peaks at $F_{\max} \approx 0.21 eE_D$ ($u \approx 1.4 v_{te}$), with the **Dreicer field**

$$E_D \equiv \frac{4\pi n_e e^3 \ln \Lambda}{T_e} = \frac{e \ln \Lambda}{\lambda_{De}^2} = 2.6 \times 10^{-11} \frac{n_e \ln \Lambda}{T_{\text{eV}}} \text{ V m}^{-1}$$

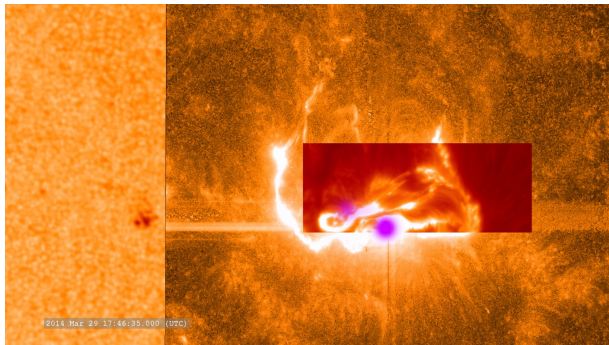
- ▶ $E \gtrsim 0.2 E_D$: the field beats the maximum friction: **bulk runaway**
- ▶ $E \ll E_D$: only the tail. Fast test electron: $F(v) \simeq (2 + Z) 4\pi n_e e^4 \ln \Lambda / m_e v^2$;
 $eE = F(v_c)$ gives

$$\frac{v_c^2}{v_{te}^2} = (2 + Z) \frac{E_D}{E}, \quad \text{fraction} \sim \exp\left[-\frac{(2 + Z)E_D}{2E}\right]$$

Flares, tokamaks, thunderstorms

Corona: $E_D \approx 5 \times 10^{-3} \text{ V/m}$, exceeded in flare current sheets (hard X-rays).

Runaway electrons in a solar flare



NASA Goddard: SDO, IRIS, Hinode, RHESSI, Sacramento Peak

The X-class flare of 29 March 2014, the best observed ever.

- ▶ reconnection electric fields far exceed $E_D \approx 5 \times 10^{-3}$ V/m
- ▶ electrons accelerated to tens of keV stream down the loops
- ▶ where they hit the dense chromosphere they brake: hard X-ray **bremsstrahlung** at the footpoints (RHESSI)

Radiation from collisions

Bremsstrahlung: who radiates?

An accelerated charge radiates the Larmor power

$$P = \frac{2e^2 a^2}{3c^3}$$

- ▶ **e - e collisions:** no dipole radiation. The dipole moment $\sum q\mathbf{r} = -e(\mathbf{r}_1 + \mathbf{r}_2) \propto$ centre of mass, which does not accelerate
- ▶ **e - i collisions:** the electron radiates; the ion's acceleration is smaller by m_e/m_i , its power by $(m_e/m_i)^2$

Key result

Bremsstrahlung (free-free emission) = radiation of electrons accelerated in the Coulomb field of ions.

Bremsstrahlung: an estimate

Ion at distance r : $a = Ze^2/(m_e r^2)$. Random ions add incoherently:

$$\langle a^2 \rangle = n_i \int_{r_{\min}}^{\infty} \left(\frac{Ze^2}{m_e r^2} \right)^2 4\pi r^2 dr = \frac{4\pi n_i Z^2 e^4}{m_e^2} \int_{r_{\min}}^{\infty} \frac{dr}{r^2} = \frac{4\pi n_i Z^2 e^4}{m_e^2 r_{\min}}$$

Dominated by **close** encounters, the opposite of momentum transfer. Cut-off: an encounter at r emits $\omega \lesssim v/r$, and $\hbar\omega \lesssim m_e v^2 \Rightarrow r_{\min} \sim \hbar/m_e v$ (de Broglie). Per electron:

$$P = \frac{2e^2}{3c^3} \cdot \frac{4\pi n_i Z^2 e^4 v}{m_e \hbar} = \frac{8\pi}{3} \frac{Z^2 e^6 n_i v}{m_e c^3 \hbar}$$

Per unit volume, $n_e P$ with $v \sim (T/m_e)^{1/2}$:

$$\varepsilon_{\text{ff}} \sim \frac{8\pi}{3} \frac{Z^2 e^6}{m_e^{3/2} c^3 \hbar} n_e n_i T^{1/2} = 1.5 \times 10^{-27} Z^2 n_e n_i T_K^{1/2} \text{ erg cm}^{-3} \text{ s}^{-1}$$

Bremsstrahlung: exact result and spectrum

Full calculation (spectrum per encounter, integrated over b , Maxwellian average):

$$\varepsilon_\nu = 6.8 \times 10^{-38} Z^2 n_e n_i T_K^{-1/2} e^{-h\nu/T} \bar{g}_{\text{ff}} \text{ erg cm}^{-3} \text{ s}^{-1} \text{ Hz}^{-1}$$

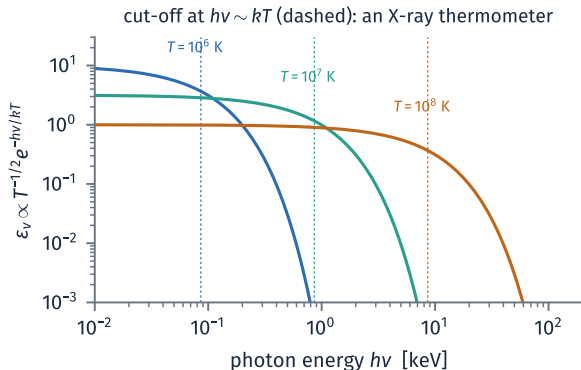
flat up to $h\nu \sim T$ (short pulses $\rightarrow$ flat Fourier spectrum); only electrons with $\frac{1}{2}m_e v^2 > h\nu$ can emit at ν , hence $e^{-h\nu/T}$. Gaunt factor $\bar{g} \approx 1$ –1.5. Integrate, $\int_0^\infty e^{-h\nu/T} d\nu = T/h$:

Key result: bremsstrahlung emissivity

$$\varepsilon_{\text{ff}} = 1.4 \times 10^{-27} Z^2 n_e n_i T_K^{1/2} \bar{g}_B \text{ erg cm}^{-3} \text{ s}^{-1}$$

The estimate was within 10%. X-ray astronomy: the **cut-off** measures T_e ; the **normalisation** measures $\int n_e n_i dV$ (emission measure) and hence n .

Thermal bremsstrahlung spectra



- ▶ flat up to $h\nu \sim kT$, then $e^{-h\nu/kT}$
- ▶ corona (10^6 K): soft X-rays, < 0.1 keV
- ▶ clusters (10^8 K): $kT \approx 9$ keV; the cut-off seen by Chandra and XMM-Newton gives the gas temperature, the normalisation its density

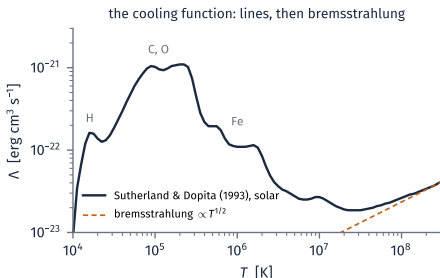
Recombination, lines and the cooling function

Also **recombination** (free-bound: kinetic + binding energy) and **lines** (collisional excitation). Optically thin, coronal equilibrium: all two-body,

$$\varepsilon = n_e n_H \Lambda(T) \quad (\text{cooling function})$$

Solar abundances:

- ▶ rises steeply above 10^4 K (H ionises)
- ▶ peak $\Lambda \approx 10^{-21}$ erg cm³/s near 10^5 K (C, O lines); Fe near 10^6 K
- ▶ $T \gtrsim 2 \times 10^7$ K: bremsstrahlung, $\Lambda \propto T^{1/2}$



data: Sutherland & Dopita (1993), ApJS 88, 253

Cooling time

Fully ionised H, $n_e = n_i = n$: thermal energy $\frac{3}{2}(n_e + n_i)T = 3nT$, loss $n^2\Lambda$:

$$t_{\text{cool}} = \frac{3nT}{n^2\Lambda(T)} = \frac{3T}{n\Lambda(T)} \xrightarrow{\text{bremsstrahlung, } \bar{g}_B \approx 1.2} t_{\text{ff}} \approx 2.5 \times 10^{11} \frac{T_K^{1/2}}{n} \text{ s}$$

$t_{\text{cool}} \propto 1/n$: cooling gas is compressed, cools faster: thermal instability.

Cooling flows in clusters

Bulk (10^{-3} cm^{-3} , 10^8 K): $t_{\text{ff}} \approx 80 \text{ Gyr}$.
Cores (10^{-2} , $3 \times 10^7 \text{ K}$): 4 Gyr , yet little
star formation: AGN heating.

The solar corona

10^9 cm^{-3} , 10^6 K , $\Lambda \approx 10^{-22}$: $t_{\text{cool}} \approx 4 \times 10^3 \text{ s}$, an hour. It must be heated continuously.

Summary

Rutherford	$\tan(\theta/2) = b_{90}/b, b_{90} = q_1 q_2 / m_r u^2$
Momentum transfer	$\sigma_m = 4\pi b_{90}^2 \ln \Lambda$
Coulomb log	$\Lambda = 12\pi n_e \lambda_{De}^3 / Z$
Collision freq.	$\nu_{ei} \propto n_e T_e^{-3/2}, \nu(v) \propto v^{-3}$
Hierarchy	$1 : (m_i/m_e)^{1/2} : m_i/m_e$
Diffusivity	$\eta \simeq 10^{13} T_K^{-3/2} \text{ cm}^2/\text{s}$
Dreicer	$E_D = e \ln \Lambda / \lambda_{De}^2$
Bremsstrahlung	$1.4 \times 10^{-27} Z^2 n_e n_i T^{1/2}$

- ▶ distant encounters: diffusion in v
- ▶ $\nu \propto v^{-3}$: runaway, collisionless hot plasmas
- ▶ η independent of n
- ▶ close encounters: bremsstrahlung

Next: module 3a, charged-particle orbits and drifts.

Problems: set 1.