

Coulomb collisions

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Outline

1. Binary Coulomb scattering
2. The Coulomb logarithm
3. Collision frequencies and relaxation

Reading

Spitzer (1962), ch. 5

Fitzpatrick, ch. 3

Rybicki & Lightman, ch. 5

Notes: chapter 2, §2.1–2.3

Problems: set 1

Collisions in a collective plasma

Chapter 1: beyond λ_D each charge is screened, fields are smooth. But the plasma is still made of point charges, which deflect one another: **Coulomb collisions**.

What collisions do

- ▶ relax $f(\mathbf{v})$ towards a Maxwellian
- ▶ exchange energy between species
- ▶ resist the current: resistivity
- ▶ make the plasma radiate

Today's message

Collisions are dominated by many *weak, distant* encounters, not by close ones. Their strength is set by

$$\ln \Lambda \approx 10-40$$

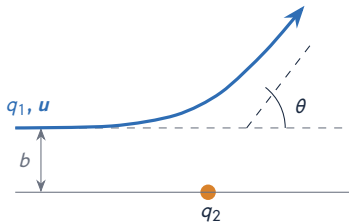
and their rate falls as v^{-3} .

Key result

The resistivity derived today sets the magnetic diffusivity η and the magnetic Reynolds number $R_m = UL/\eta$: the control parameter of dynamo theory.

Binary Coulomb scattering

The two-body problem



Centre of mass moves uniformly; the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ obeys

$$m_r \ddot{\mathbf{r}} = \frac{q_1 q_2}{r^2} \hat{\mathbf{r}}, \quad m_r = \frac{m_1 m_2}{m_1 + m_2}$$

one particle of reduced mass m_r in a fixed Coulomb field. Speed u unchanged, direction turned by θ .

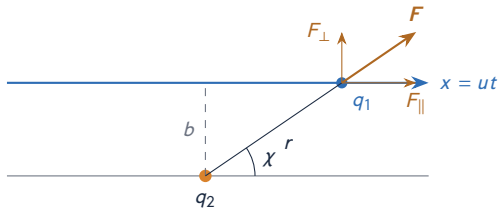
Natural scale: potential energy = kinetic energy at distance

Key result

$$b_{90} \equiv \frac{q_1 q_2}{m_r u^2} \quad (e-i: m_r \simeq m_e; \text{thermal: } b_{90} \sim Z e^2 / 3T)$$

Small-angle deflection (1): geometry

For $b \gg b_{90}$ the path is hardly bent: integrate the force along the *undeflected* straight line $y = b$, $x = ut$ (impulse approximation).



$r = \sqrt{b^2 + u^2 t^2}$, force $F = q_1 q_2 / r^2$ along $\hat{r} = (ut/r, b/r)$:

$$F_{\parallel} = \frac{q_1 q_2}{r^2} \frac{ut}{r}, \quad F_{\perp} = \frac{q_1 q_2}{r^2} \underbrace{\frac{b}{r}}_{\sin \chi}$$

$F_{\parallel}$ is odd in t : slows the particle on the way in, speeds it up on the way out. **Only $F_{\perp}$ accumulates.**

$$\Delta p_{\perp} = \int_{-\infty}^{\infty} F_{\perp} dt = \int_{-\infty}^{\infty} \frac{q_1 q_2}{r^2} \frac{b}{r} dt = q_1 q_2 b \int_{-\infty}^{\infty} \frac{dt}{(b^2 + u^2 t^2)^{3/2}}$$

Small-angle deflection (2): the integral

$$\Delta p_{\perp} = q_1 q_2 b \int_{-\infty}^{\infty} \frac{dt}{(b^2 + u^2 t^2)^{3/2}}$$

Substitute $ut = b \tan \psi$, so $dt = (b/u) \sec^2 \psi d\psi$ and $b^2 + u^2 t^2 = b^2 \sec^2 \psi$:

$$\int_{-\infty}^{\infty} \frac{dt}{(b^2 + u^2 t^2)^{3/2}} = \int_{-\pi/2}^{\pi/2} \frac{(b/u) \sec^2 \psi}{b^3 \sec^3 \psi} d\psi = \frac{1}{ub^2} \int_{-\pi/2}^{\pi/2} \cos \psi d\psi = \frac{2}{ub^2}$$

$$\Rightarrow \Delta p_{\perp} = \frac{2q_1 q_2}{bu}$$

Reading: the force $q_1 q_2 / b^2$ acts for a time $\sim 2b/u$.

Key result

$$\theta \simeq \frac{\Delta p_{\perp}}{m_r u} = \frac{2q_1 q_2}{m_r u^2 b} = \frac{2b_{90}}{b}$$

Exact orbit: Rutherford

Binet equation for $s = 1/r$, with $L = m_r u b$ and repulsive force $F = q_1 q_2 s^2$:

$$\frac{d^2 s}{d\varphi^2} + s = -\frac{m_r F}{L^2 s^2} = -\frac{q_1 q_2}{m_r u^2 b^2} = -\frac{b_{90}}{b^2}$$

Incoming from $\varphi = 0$: $s(0) = 0$, $s'(0) = 1/b$ (far away $r \sin \varphi \simeq b$). Solution:

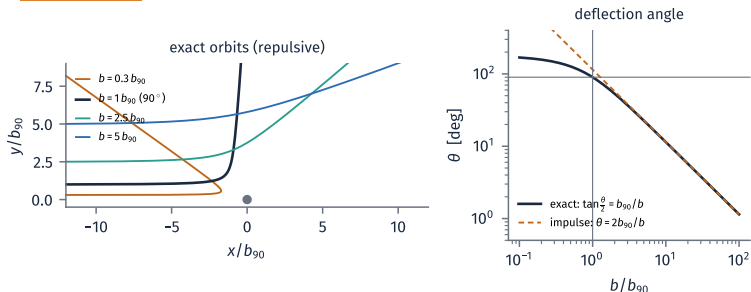
$$s(\varphi) = \frac{1}{b} \sin \varphi + \frac{b_{90}}{b^2} (\cos \varphi - 1)$$

Outgoing asymptote, $s(\varphi_f) = 0$: $\frac{1 - \cos \varphi_f}{\sin \varphi_f} = \tan \frac{\varphi_f}{2} = \frac{b}{b_{90}}$. With $\theta = \pi - \varphi_f$:

Key result: Rutherford deflection

$$\tan \frac{\theta}{2} = \frac{b_{90}}{b} \quad (b = b_{90}: 90^\circ; b \gg b_{90}: \theta \simeq 2b_{90}/b)$$

Rutherford orbits, computed



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- ▶ exact orbits integrated numerically: $b = b_{90}$ turns by 90° , $b < b_{90}$ bounces back
- ▶ the impulse formula $\theta = 2b_{90}/b$ is excellent for $b \gtrsim 2b_{90}$, which is where almost all plasma collisions happen
- ▶ 1911: Rutherford explained the alpha particles of Geiger and Marsden bouncing back from gold foil with this formula, and discovered the nucleus

The Rutherford cross section

Impact parameters in $[b, b + db]$: area $d\sigma = 2\pi b db$, scattered into $d\Omega = 2\pi \sin \theta d\theta$. From $b = b_{90} \cot(\theta/2)$: $|db/d\theta| = b_{90}/[2 \sin^2(\theta/2)]$, so

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right| = \frac{b_{90} \cot(\theta/2)}{2 \sin(\theta/2) \cos(\theta/2)} \cdot \frac{b_{90}}{2 \sin^2(\theta/2)}$$

Key result: Rutherford (1911)

$$\frac{d\sigma}{d\Omega} = \frac{b_{90}^2}{4 \sin^4(\theta/2)}$$

- ▶ diverges as θ^{-4} for $\theta \rightarrow 0$: the total cross section is **infinite**
- ▶ every particle, however distant, is deflected a little: long-range force

The momentum-transfer cross section

Forward component of the relative velocity: $u \rightarrow u \cos \theta$; fraction lost $1 - \cos \theta$:

$$\sigma_m \equiv \int (1 - \cos \theta) d\sigma = \int_0^{b_{\max}} (1 - \cos \theta) 2\pi b db$$

Express the weight through b : $1 - \cos \theta = 2 \sin^2 \frac{\theta}{2} = \frac{2}{1 + \cot^2(\theta/2)} = \frac{2b_{90}^2}{b^2 + b_{90}^2}$

$$\sigma_m = 4\pi b_{90}^2 \int_0^{b_{\max}} \frac{b db}{b^2 + b_{90}^2} = 2\pi b_{90}^2 \ln \left(1 + \frac{b_{\max}^2}{b_{90}^2} \right)$$

Key result: still divergent, but only logarithmically

$$\sigma_m \simeq 4\pi b_{90}^2 \ln \frac{b_{\max}}{b_{90}}$$

The Coulomb logarithm

Cutting off the divergence

Upper cut-off: screening

Beyond λ_D the potential is $q e^{-r/\lambda_D}/r$ (module 1a): no deflection.

$$b_{\max} = \lambda_D$$

Lower cut-off: classical or quantum

b_{90} (large angles, already in the exact σ_m), or the de Broglie wavelength

$$b_q = \frac{\hbar}{2m_r u}$$

$$b_{\min} = \max(b_{90}, b_q)$$

Which one? For an electron on an ion of charge Ze :

$$\frac{b_{90}}{b_q} = \frac{2Ze^2}{\hbar u} = 2Z \frac{\alpha c}{u}, \quad \alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}$$

Slow electrons ($u < 2Z\alpha c$): classical. Thermal electrons: quantum for $T \gtrsim 10 Z^2$ eV.

The thermal Coulomb logarithm

Classical cut-off, $m_r u^2 \rightarrow 3T$: $b_{\min} = Ze^2/3T$. Use $T = 4\pi n_e e^2 \lambda_{De}^2$:

$$\Lambda = \frac{\lambda_{De}}{b_{\min}} = \frac{3T\lambda_{De}}{Ze^2} = \frac{3 \cdot 4\pi n_e e^2 \lambda_{De}^2 \cdot \lambda_{De}}{Ze^2} = \frac{12\pi n_e \lambda_{De}^3}{Z} = \frac{9N_D}{Z}$$

The plasma parameter of chapter 1 returns as the ratio of the largest to the smallest impact parameter.

Key result: NRL, n_e in cm^{-3} , T_e in eV

$$\ln \Lambda \simeq 23 - \ln(n_e^{1/2} Z T_e^{-3/2})$$

$$T_e \lesssim 10Z^2 \text{ eV}$$

$$\ln \Lambda \simeq 24 - \ln(n_e^{1/2} T_e^{-1})$$

$$T_e \gtrsim 10Z^2 \text{ eV (quantum)}$$

	$\ln \Lambda$
Intracluster medium	36
Warm ionised ISM	24
Solar corona	18
Tokamak	17
Solar core	≈ 2

Distant encounters win: the random-walk picture

Per unit time, $2\pi b db nu$ encounters at impact parameter b , each turning the particle by $\theta = 2b_{90}/b$ in a *random* direction; these add in quadrature:

$$\frac{d\langle\theta^2\rangle}{dt} = nu \int_{b_{\min}}^{\lambda_D} \left(\frac{2b_{90}}{b}\right)^2 2\pi b db = 8\pi nu b_{90}^2 \int_{b_{\min}}^{\lambda_D} \frac{db}{b} = 8\pi nu b_{90}^2 \ln \Lambda$$

- ▶ every **decade** of b contributes equally:
hence the logarithm

- ▶ $\langle\theta^2\rangle \sim 1$ after a time $1/\nu_{90}$,
 $\nu_{90} = 8\pi nu b_{90}^2 \ln \Lambda$

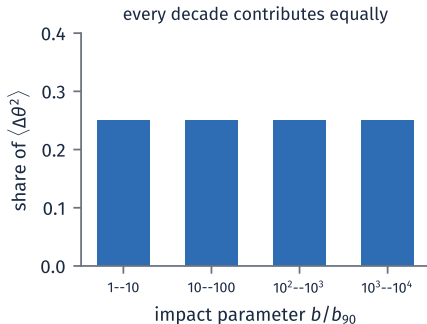
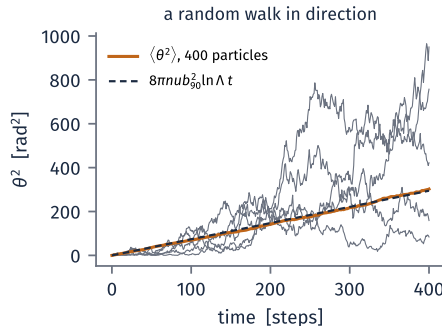
Compare single large-angle collisions,
rate $n\pi b_{90}^2 u$:

$$\frac{\nu_{90}}{n\pi b_{90}^2 u} = 8 \ln \Lambda \approx 100\text{--}300$$

Key result

Coulomb collisions are a **diffusion in velocity space** (Fokker–Planck, ch. 7).

The random walk, computed



Monte Carlo of encounters drawn from $2\pi b db$, deflection $2b_{90}/b$ in random directions:

- ▶ left: each decade of b gives **the same** share of $\langle \Delta\theta^2 \rangle$: hence $\ln \Lambda$
- ▶ right: single particles wander (grey); the mean square deflection grows linearly, as $8\pi n u b_{90}^2 \ln \Lambda t$: diffusion in velocity space

Collision frequencies and relaxation

The collision frequency of a test electron

Each collision removes momentum $m_r u(1 - \cos \theta)$ from particle a . Rate of momentum loss:

$$\frac{dp_a}{dt} = -n_b u \int m_r u(1 - \cos \theta) d\sigma = -m_r n_b u^2 \sigma_m \equiv -v_{ab} m_a u$$

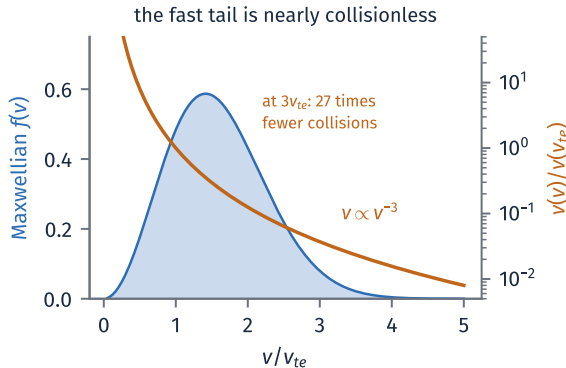
Electron on ions of charge Ze : $m_r \simeq m_e$, $u = v$, $b_{90} = Ze^2/m_e v^2$, and $n_i Z^2 = n_e Z$:

$$\nu_{ei}(v) = n_i 4\pi \left(\frac{Ze^2}{m_e v^2} \right)^2 \ln \Lambda v$$

Key result: fast particles barely collide

$$\nu_{ei}(v) = \frac{4\pi n_e Z e^4 \ln \Lambda}{m_e^2 v^3} \quad (v \text{ from the flux, } v^{-4} \text{ from } b_{90}^2)$$

Fast particles barely collide



- ▶ the bulk of a Maxwellian sits near $v \approx 1.4 v_{te}$
- ▶ the tail, $v \gtrsim 3v_{te}$, collides 30–100 times less often
- ▶ hence: **runaway electrons** (module 2b), hot plasmas that are nearly collisionless, and the Spitzer correction to the resistivity

Thermal collision frequencies

Put $\nu \sim \nu_{te} = (T_e/m_e)^{1/2}$: $\nu_{ei} \sim 4\pi n_e Z e^4 \ln \Lambda / (m_e^{1/2} T_e^{3/2})$.

The Maxwellian average (ch. 7, and checked below) replaces 4π by $4\sqrt{2}\pi/3 \approx 3.34$.

Key result: electron-ion collision frequency

$$\nu_{ei} = \frac{4\sqrt{2}\pi}{3} \frac{n_e Z e^4 \ln \Lambda}{m_e^{1/2} T_e^{3/2}} = 2.91 \times 10^{-6} \frac{Z n_e \ln \Lambda}{T_{\text{eV}}^{3/2}} \text{ s}^{-1}$$

- ▶ $\nu \propto n T^{-3/2}$: at 10^8 K a plasma collides 10^6 times less than at 10^4 K
- ▶ $\nu_{ei}/\omega_{pe} \approx \ln \Lambda / (10 N_D) \ll 1$: many oscillations per collision
- ▶ like particles: $\nu_{ee} \sim \nu_{ei}$; ions: same formula with m_i, T_i ,

$$\frac{\nu_{ii}}{\nu_{ee}} \sim \left(\frac{m_e}{m_i} \right)^{1/2} \left(\frac{T_e}{T_i} \right)^{3/2} \approx \frac{1}{43}$$

Energy exchange: ping-pong balls on a bowling ball

Light particle m , speed v , hits heavy $M \gg m$ at rest. Speed almost unchanged, direction turned by θ :

$$|\Delta \mathbf{p}|^2 = |m\mathbf{v}' - m\mathbf{v}|^2 = 2m^2v^2(1 - \cos \theta)$$

Energy given to M :

$$\Delta E = \frac{|\Delta \mathbf{p}|^2}{2M} = \frac{m}{M} mv^2(1 - \cos \theta) = \frac{2m}{M} E (1 - \cos \theta)$$

Same angular weight as momentum transfer, so

Key result

$$v_{ei}^E \simeq \frac{2m_e}{m_i} v_{ei}$$

The relaxation hierarchy

process	rate	relative time
electrons become Maxwellian	$\nu_{ee} \sim \nu_{ei}$	1
ions become Maxwellian	$\nu_{ii} \sim (m_e/m_i)^{1/2} \nu_{ee}$	≈ 43
T_e and T_i equalise	$\nu_{eq} \sim (m_e/m_i) \nu_{ei}$	≈ 1836

$$\frac{dT_e}{dt} = -\nu_{eq}(T_e - T_i), \quad \nu_{eq} = \frac{2m_e}{m_i} \nu_{ei} = 3.2 \times 10^{-9} \frac{Z^2 n_e \ln \Lambda}{A T_{\text{eV}}^{3/2}} \text{ s}^{-1}$$

Two-temperature plasmas wherever heating acts on one species faster than ν_{eq} : behind collisionless shocks (ions heated, ch. 14), in the solar wind, in hot accretion flows onto black holes ($T_i \sim 10 T_e$).

The mean free path

Distance travelled at v_{te} in one collision time:

$$\lambda_{\text{mfp}} = \frac{v_{te}}{v_{ei}} = \left(\frac{T_e}{m_e} \right)^{1/2} \cdot \frac{3 m_e^{1/2} T_e^{3/2}}{4\sqrt{2\pi} n_e Z e^4 \ln \Lambda} = \frac{3}{4\sqrt{2\pi}} \frac{T_e^2}{n_e Z e^4 \ln \Lambda}$$

Key result

$$\lambda_{\text{mfp}} = 1.44 \times 10^{13} \frac{T_{\text{eV}}^2}{Z n_e \ln \Lambda} \text{ cm}$$

- ▶ the mass cancels: electrons and ions at equal T have the same mean free path
- ▶ $\lambda_{\text{mfp}} \propto T^2/n$: hot, dilute plasmas have enormous mean free paths

Collisional or collisionless?

Compare λ_{mfp} with the size L of the system:

	mean free path λ_{mfp}	system size L	
Solar corona	50 km	10^4 – 10^5 km	collisional along loops
Solar wind, 1 AU	~ 0.5 AU	1 AU	marginal
Tokamak	~ 5 km	~ 10 m	collisionless along $\mathbf{B}$
Intracluster gas	~ 10 kpc	~ 1 Mpc	weakly collisional

- ▶ $\lambda_{\text{mfp}} \ll L$: collisions keep f Maxwellian, fluid description (MHD, ch. 9–15)
- ▶ $\lambda_{\text{mfp}} \gtrsim L$: kinetic theory needed (ch. 5–8)
- ▶ across $\mathbf{B}$ the gyroradius, not λ_{mfp} , is the step length: even a collisionless plasma can behave as a fluid perpendicular to the field

So far, and next

In this module

- ▶ Rutherford: $\tan(\theta/2) = b_{90}/b$; momentum transfer $\sigma_m = 4\pi b_{90}^2 \ln \Lambda$
- ▶ distant encounters dominate:
 $\ln \Lambda = \ln(12\pi n \lambda_{De}^3) \approx 10-40$
- ▶ $\nu_{ei} \propto nT^{-3/2}$, $\nu(\nu) \propto \nu^{-3}$; hierarchy
1 : 43 : 1836

Next: module 2b: resistivity, magnetic diffusivity, runaway electrons and bremsstrahlung.