

The plasma state: regimes and ionisation

Alfio Bonanno

INAF – Osservatorio Astrofisico di Catania

Outline

1. Plasma parameter and regimes
2. Ionisation equilibrium: the Saha equation

Reading

Boyd & Sanderson, ch. 1

Bittencourt, ch. 1

Fitzpatrick, ch. 1

Zel'dovich & Raizer, ch. III

Notes: chapter 1, §1.4–1.5

Problems: set 1

Where we are

Key result: recap

- ▶ Debye length $\lambda_D = (T/4\pi n e^2)^{1/2}$: screening, quasi-neutrality
- ▶ plasma frequency $\omega_{pe} = (4\pi n e^2/m_e)^{1/2}$, $\omega_{pe}\lambda_D = v_{te}$
- ▶ correlation energy $|w|/T = 1/12N_D$: ideal when $N_D \gg 1$
- ▶ today: when is a plasma ideal, classical, ionised?

Plasma parameter and regimes

Coupling: potential vs kinetic energy

Mean interparticle distance (Wigner–Seitz radius) $a = (3/4\pi n)^{1/3}$:

$$\Gamma \equiv \frac{\text{Coulomb energy of neighbours}}{\text{thermal energy}} = \frac{Z^2 e^2}{a T}$$

For $Z = 1$, use $\lambda_D^2 = T/4\pi n e^2$ and $a^3 = 3/4\pi n$:

$$N_D^2 = \left(\frac{4\pi}{3}\right)^2 n^2 \lambda_D^6 = \frac{T^3}{36\pi n e^6}, \quad \Gamma^3 = \frac{e^6}{a^3 T^3} = \frac{4\pi n e^6}{3 T^3} \Rightarrow N_D^2 \Gamma^3 = \frac{1}{27}$$

Key result: coupling parameter

$$N_D = \frac{1}{3\sqrt{3} \Gamma^{3/2}}$$

Weak and strong coupling

$\Gamma \ll 1, N_D \gg 1$: weakly coupled

- ▶ many particles screen each charge
- ▶ collective, smooth fields; ideal-gas EOS
- ▶ small-angle collisions dominate

$N_D \propto T^{3/2}/n^{1/2}$: hot and dilute is ideal. Conventions differ: $n\lambda_D^3$, N_D , or $\Lambda = 12\pi n\lambda_D^3$ in $\ln \Lambda$.

$\Gamma \gtrsim 1$: strongly coupled

- ▶ correlations like a liquid
- ▶ Debye theory fails
- ▶ crystallisation at $\Gamma \approx 175$ (one-component plasma)

Quantum degeneracy

Quantum effects matter when the thermal de Broglie wavelength reaches the spacing (electrons long before ions):

$$n\lambda_{\text{th}}^3 \gtrsim 1, \quad \lambda_{\text{th}} = \frac{h}{(2\pi m_e T)^{1/2}} \propto m^{-1/2}$$

At $T = 0$, electrons fill momentum space up to p_F , two per cell $(2\pi\hbar)^3$:

$$n_e = \frac{2}{(2\pi\hbar)^3} \cdot \frac{4\pi}{3} p_F^3 \Rightarrow p_F = \hbar(3\pi^2 n_e)^{1/3}$$

Key result: Fermi temperature

$$T_F = \frac{p_F^2}{2m_e} = \frac{\hbar^2(3\pi^2 n_e)^{2/3}}{2m_e} = 4.23 \times 10^{-11} \left(\frac{n_e}{\text{cm}^{-3}} \right)^{2/3} \text{ K}$$

Coupling in a degenerate plasma

Classical for $T \gg T_F$. When $T \ll T_F$ the kinetic energy is $\sim E_F$, not T . Since $n^{2/3} \sim a^{-2}$, $E_F \sim \hbar^2/m_e a^2$:

$$\frac{e^2/a}{E_F} \sim \frac{e^2/a}{\hbar^2/m_e a^2} = \frac{a}{a_B} \equiv r_s, \quad a_B = \frac{\hbar^2}{m_e e^2} = 0.53 \text{ \AA}$$

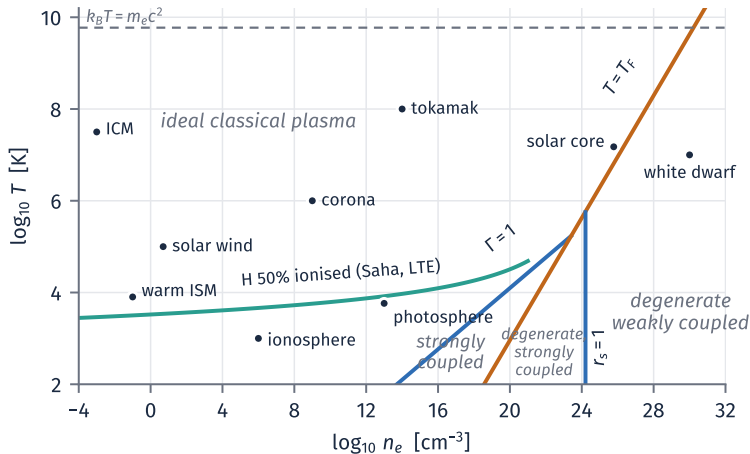
Key result

Degenerate plasmas become **more ideal** as they are compressed:

$$r_s < 1 \text{ for } n_e > 3/4\pi a_B^3 = 1.6 \times 10^{24} \text{ cm}^{-3}$$

- ▶ classical plasma: $\Gamma \propto n^{1/3}/T$, coupling grows with density
- ▶ degenerate plasma: $r_s \propto n^{-1/3}$, coupling falls with density
- ▶ electrons in white dwarfs: degenerate and ideal; ions: classical and strongly coupled

The T - n diagram



blue $\Gamma = 1$, then $r_s = 1$

orange $T = T_F$

teal H half-ionised (Saha, LTE)

Two dense plasmas

The solar core

$$n_e = 6 \times 10^{25} \text{ cm}^{-3}, T = 1.5 \times 10^7 \text{ K}$$

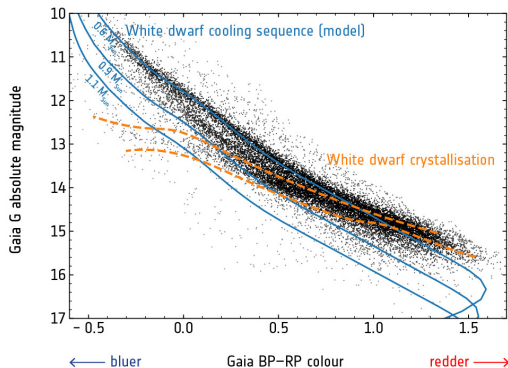
- ▶ $\Gamma \approx 0.07$, $N_D \approx 10$: weakly but not very weakly coupled
- ▶ $T/T_F \approx 2$, $n\lambda_{\text{th}}^3 \approx 0.4$: mildly degenerate
- ▶ corrections to the ideal-gas EOS at the per-cent level, measured by helioseismology

A white-dwarf interior

$$n_e \sim 10^{30} \text{ cm}^{-3}, T \sim 10^7 \text{ K, carbon-oxygen}$$

- ▶ electrons: $T/T_F \sim 10^{-3}$, $r_s \approx 0.01$: degenerate, *ideal*
- ▶ C ions: $\Gamma_i = Z^2 e^2 / a_i T \approx 50$: strongly coupled
- ▶ as the star cools, $\Gamma_i \rightarrow 175$ and the core crystallises; latent heat delays cooling (seen in the Gaia white-dwarf sequence)

White dwarfs crystallise



ESA; courtesy of P.-E. Tremblay et al. (2019)

- ▶ 15 000 white dwarfs within 100 pc (Gaia DR2)
- ▶ a pile-up along the cooling sequence (orange): stars linger there
- ▶ the ion plasma reaches $\Gamma_i \approx 175$ and freezes into a crystal; latent heat delays the cooling
- ▶ a phase transition of a strongly coupled plasma, predicted in the 1960s, seen in a colour-magnitude diagram

Ionisation equilibrium: the Saha equation

A puzzle

Hydrogen has $I = 13.6$ eV, i.e. $I/k_B = 158\,000$ K.

Yet the solar photosphere (5800 K) is weakly ionised, and hydrogen is half ionised at $\approx 10^4$ K in stellar atmospheres and at ≈ 3700 K in the early Universe.

Key result: entropy wins

A bound electron occupies one state; a free electron can occupy any of $\sim 1/(n\lambda_{\text{th}}^3) \gg 1$ states. Ionisation is favoured once

$$e^{-I/T} \gtrsim n\lambda_{\text{th}}^3 \quad \Rightarrow \quad T_{\text{ion}} \approx \frac{I}{\ln(1/n\lambda_{\text{th}}^3)} \sim \frac{I}{20-40}$$

The more dilute the gas, the lower the ionisation temperature.

Derivation (1): ideal gas with internal states

One particle of species s : translational $\times$ internal states (weights U_s , ground-state energy E_s):

$$Z_{1,s} = \underbrace{\int \frac{d^3x d^3p}{h^3} e^{-p^2/2m_s T}}_{V (2\pi m_s T)^{3/2} / h^3} \times U_s e^{-E_s/T} = \frac{V U_s}{\lambda_s^3} e^{-E_s/T}, \quad \lambda_s = \frac{h}{(2\pi m_s T)^{1/2}}$$

N_s identical, non-interacting particles: $Z = Z_{1,s}^{N_s} / N_s!$; Stirling $\ln N! \simeq N \ln N - N$:

$$F = -T \ln Z = -N_s T \left[\ln \frac{Z_{1,s}}{N_s} + 1 \right]$$

$$\mu_s = \left. \frac{\partial F}{\partial N_s} \right|_{T,V} = -T \ln \frac{Z_{1,s}}{N_s} \implies n_s = \frac{N_s}{V} = U_s \left(\frac{2\pi m_s T}{h^2} \right)^{3/2} e^{(\mu_s - E_s)/T}$$

Derivation (2): chemical equilibrium

Reaction $\text{H} \rightleftharpoons p + e$ at fixed T, V : F is minimum, with $dN_p = dN_e = -dN_H$:

$$dF = \mu_p dN_p + \mu_e dN_e + \mu_H dN_H = 0 \quad \Rightarrow \quad \mu_H = \mu_p + \mu_e$$

Form $n_p n_e / n_H$ with n_s from the previous slide:

$$\frac{n_p n_e}{n_H} = \frac{U_p U_e}{U_H} \left(\frac{2\pi T}{h^2} \right)^{3/2} \left(\frac{m_p m_e}{m_H} \right)^{3/2} e^{(\mu_p + \mu_e - \mu_H)/T} e^{-(E_p + E_e - E_H)/T}$$

- ▶ $\mu_p + \mu_e - \mu_H = 0$: the unknown chemical potentials drop out
- ▶ $m_p \simeq m_H$: only the electron mass survives
- ▶ $E_p + E_e - E_H = I$, the ionisation energy

The Saha equation

$$\frac{n_p n_e}{n_H} = \frac{U_p U_e}{U_H} \left(\frac{2\pi m_e T}{h^2} \right)^{3/2} e^{-I/T}, \quad U_e = 2, U_p = 1, U_H \simeq 2$$

U_e : electron spin. U_H : 1s ground state, two spin states; nuclear spin cancels.

Key result: Saha (1920), any element, stage $i \rightarrow i + 1$

$$\frac{n_{i+1} n_e}{n_i} = 2 \frac{U_{i+1}}{U_i} \left(\frac{2\pi m_e T}{h^2} \right)^{3/2} e^{-I_i/T} \equiv K_i(T)$$

- ▶ $(2\pi m_e T / h^2)^{3/2} = \lambda_{\text{th}}^{-3}$: the quantum concentration of free electrons; the puzzle made quantitative
- ▶ one Saha equation per ionisation stage
- ▶ requires **local thermodynamic equilibrium** and an **ideal gas**

Saha for pure hydrogen

Total density $n = n_{\text{H}} + n_p$, ionisation fraction $x = n_p/n$, neutrality $n_e = n_p$:

$$n_p = n_e = xn, \quad n_{\text{H}} = (1 - x)n$$

$$\frac{(xn)^2}{(1-x)n} = K(T) \quad \Rightarrow \quad \frac{x^2}{1-x} = \frac{K(T)}{n} = \frac{1}{n\lambda_{\text{th}}^3} e^{-I/T}$$

Numbers: $(2\pi m_e k_B / h^2)^{3/2} = 2.415 \times 10^{15} \text{ cm}^{-3} \text{ K}^{-3/2}$, $I/k_B = 157\,900 \text{ K}$:

Key result

$$\frac{x^2}{1-x} = \frac{2.415 \times 10^{15}}{n [\text{cm}^{-3}]} T_{\text{K}}^{3/2} \exp\left(-\frac{157\,900}{T_{\text{K}}}\right)$$

Solving Saha: the half-ionisation temperature

Call $r = K/n$. Then $x^2 + rx - r = 0$, with physical root

$$x = \frac{-r + \sqrt{r^2 + 4r}}{2} = \frac{2}{1 + \sqrt{1 + 4/r}}$$

the second form avoids cancellation when $r \gg 1$

Half ionisation: $x = 1/2 \Rightarrow r = 1/2 \Rightarrow n = 2K(T_{1/2})$. Take the log:

$$\ln n = \ln(2 \times 2.415 \times 10^{15} T_{1/2}^{3/2}) - \frac{I}{T_{1/2}} \quad \Rightarrow \quad T_{1/2} = \frac{I}{\ln[4.83 \times 10^{15} T_{1/2}^{3/2} / n]}$$

- ▶ solve by iteration: start from $T_{1/2} = 10^4$ K, converges in 2-3 steps
- ▶ $I/T_{1/2}$ is the logarithm of a huge number, ≈ 15 -50
- ▶ the lower the density, the larger the log, the lower $T_{1/2}$

Solving Saha: how sharp is the transition?

Differentiate $\ln[x^2/(1-x)] = \ln r$, with $r \propto T^{3/2} e^{-I/T}$:

$$\left(\frac{2}{x} + \frac{1}{1-x}\right) dx = d \ln r = \left(\frac{3}{2} + \frac{I}{T}\right) d \ln T$$

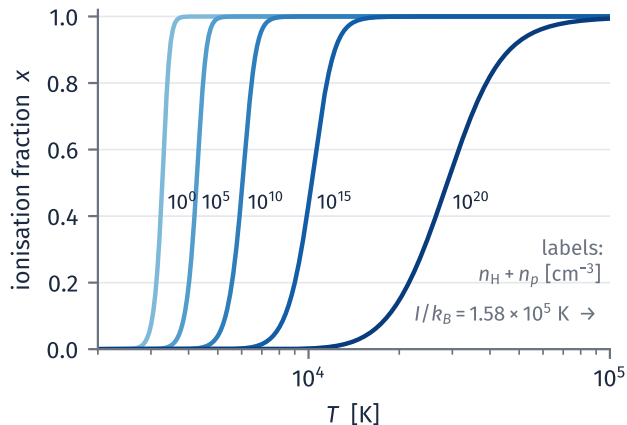
At $x = 1/2$ the bracket on the left is $4 + 2 = 6$:

$$\left. \frac{dx}{d \ln T} \right|_{x=1/2} = \frac{1}{6} \left(\frac{3}{2} + \frac{I}{T_{1/2}} \right) \approx 3-8$$

Key result

x goes from 0.1 to 0.9 over $\Delta T/T \approx 10\text{--}20\%$: ionisation is **sharp**, because T sits in an exponent with a large coefficient I/T .

Ionisation curves of hydrogen



$n_H + n_p$ [cm^{-3}]	$T_{1/2}$ [K]
1	3 300
10^5	4 300
10^{10}	6 000
10^{15}	10 200
10^{20}	28 900

The transition is sharp ($\Delta T/T \sim 10\%$) and always far below I/k_B .

Saha at work

The solar photosphere

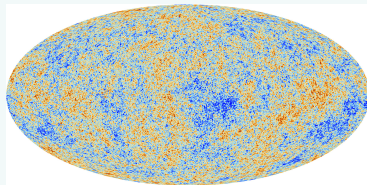
$n_{\text{H}} \approx 10^{17} \text{ cm}^{-3}$, $T = 5800 \text{ K}$:

$$x_{\text{H}} \approx 10^{-4}$$

Metals with $I \approx 5\text{--}8 \text{ eV}$ (Na, Mg, Fe, ...) supply most free electrons in the upper photosphere. These electrons form H^- , the main source of continuum opacity in the visible.

Cosmological recombination

$\eta = n_b/n_\gamma \approx 6 \times 10^{-10}$: a tiny effective density. Saha: $x = 1/2$ at $T \approx 3700 \text{ K}$ ($z \approx 1400$); $\text{Ly}\alpha$ re-ionisation delays last scattering to $z \approx 1100$ (Peebles 1968).



ESA and the Planck Collaboration

Limits of Saha I: the partition function diverges

Level n of hydrogen: energy $-I/n^2$, statistical weight $2n^2$:

$$U_H = \sum_{n=1}^{\infty} 2n^2 e^{I/n^2 T} \xrightarrow{n \rightarrow \infty} \sum 2n^2 = \infty$$

- ▶ an isolated atom has infinitely many bound states, of radius $n^2 a_B$
- ▶ in a plasma, orbits larger than $\min(a, \lambda_D)$ are destroyed by the neighbours
- ▶ truncate the sum at $n_{\max}^2 a_B \sim \min(a, \lambda_D)$: at low density $U_H \simeq 2$ is fine, at high density the truncation matters

Limits of Saha I: continuum lowering

Near an ion of charge Ze the potential is screened. For $r \ll \lambda_D$:

$$\phi(r) = \frac{Ze}{r} e^{-r/\lambda_D} \simeq \frac{Ze}{r} \left(1 - \frac{r}{\lambda_D}\right) = \frac{Ze}{r} - \frac{Ze}{\lambda_D}$$

A bound electron sits in the Coulomb well shifted up by Ze/λ_D . It becomes free at energy $-Ze^2/\lambda_D$ instead of 0:

Key result

$$I_{\text{eff}} = I - \Delta I, \quad \Delta I \simeq \frac{Ze^2}{\lambda_D}$$

- ▶ low-density limit of the Stewart–Pyatt theory
- ▶ $\Delta I/T = Ze^2/\lambda_D T \sim 1/N_D$: negligible in ideal plasmas, large in dense ones

Limits of Saha I: the solar core

Why naive Saha fails in the solar core

Saha with $n = 6 \times 10^{25} \text{ cm}^{-3}$, $T = 1.5 \times 10^7 \text{ K}$ gives $x \approx 0.75$, a quarter of hydrogen still neutral. But:

- ▶ interparticle spacing $a \approx 1.6 \times 10^{-9} \text{ cm} < a_B = 5.3 \times 10^{-9} \text{ cm}$
- ▶ a Yukawa potential $e^{-r/\lambda_D}/r$ has no bound state if $\lambda_D < 0.84 a_B$; here $\lambda_{De} \approx 0.65 a_B$, and $0.41 a_B$ with ions
- ▶ continuum lowering $e^2/\lambda_D \approx 65 \text{ eV} \gg I = 13.6 \text{ eV}$

The core is fully ionised by **pressure ionisation**, which an ideal-gas Saha equation cannot describe.

Limits of Saha II: departures from LTE

Without LTE, follow the elementary processes:

$$\frac{dn_e}{dt} = \underbrace{n_e n_H S(T)}_{\text{collisional ion.}} - \underbrace{n_e^2 n_p \beta(T)}_{\text{3-body rec.}} + \underbrace{n_H \Gamma_{\text{ph}}}_{\text{photo-ion.}} - \underbrace{n_e n_p \alpha_{\text{rr}}(T)}_{\text{radiative rec.}}$$

LTE: each process balances its inverse (detailed balance):

$$n_e n_H S = n_e^2 n_p \beta \Rightarrow \frac{n_p n_e}{n_H} = \frac{S}{\beta} = K(T) \Rightarrow \beta(T) = \frac{S(T)}{K(T)}$$

The hard rate (three-body) follows from the easy one plus Saha. Likewise Γ_{ph} in a blackbody field balances α_{rr} .

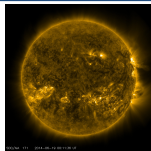
Limits of Saha II: coronal equilibrium

Hot, dilute, optically thin: no photo-ionisation, no three-body recombination.
Collisional ionisation balances radiative recombination:

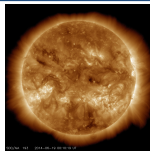
$$n_e n_H S = n_e n_p \alpha_{rr} \quad \Rightarrow \quad \frac{n_p}{n_H} = \frac{x}{1-x} = \frac{S(T)}{\alpha_{rr}(T)}$$

Key result

n_e cancels: the ionisation balance depends on T only, not on density.



171 Å, Fe IX, 0.7 MK



193 Å, Fe XII, 1.5 MK

Coronal thermometers. Each ion lives in a narrow T range, so each line images plasma at one temperature: same Sun, same hour, different structures.

Courtesy of NASA/SDO and the AIA science team, 2014-06-19

Summary

Debye length	$\lambda_{De} = (T_e/4\pi n_e e^2)^{1/2}$
Plasma frequency	$\omega_{pe} = (4\pi n_e e^2/m_e)^{1/2}$
Their product	$\omega_{pe}\lambda_{De} = v_{te}$
Plasma parameter	$N_D = \frac{4\pi}{3} n \lambda_D^3 = 1/(3\sqrt{3} \Gamma^{3/2})$
Degeneracy	$T_F = \hbar^2 (3\pi^2 n_e)^{2/3} / 2m_e$
Saha	$\frac{n_{i+1} n_e}{n_i} = 2 \frac{U_{i+1}}{U_i} \lambda_{th}^{-3} e^{-I_i/T}$

A plasma needs

- ▶ $L \gg \lambda_D$
- ▶ $N_D \gg 1$
- ▶ $\omega_{pe}\tau_n \gg 1$

Ionisation sets in at $T \sim I/30$, because of phase-space entropy; Saha needs LTE and an ideal gas.

Next: module 2a, Coulomb collisions, $\ln \Lambda$, resistivity and bremsstrahlung.

Problems: set 1.